Homotopy Models of Intensional Type Theory

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Abstract

We show how to provide a semantics for the identity type of intensional Martin-Löf type theory using the machinery of abstract homotopy theory. One specific consequence is that the identity type is soundly modelled in any Quillen model category. We then turn to the study of cocategory and (strict) interval objects and isolate two conditions under which an interval will model intensional identity types.

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Introduction

The intensional form of Martin-Löf's type theory [14] was originally intended to provide a (logical) foundation for the development of Bishop-style constructive mathematics. There are two features of this type theory which are of particular interest. Namely, the presence of *universes* and the treatment of *identity types*. It is this latter feature with which we will be presently concerned.

Recall that one of the requirements laid down by Bishop [2] for the construction of a set A is that it must come equipped with an equivalence relation $=_A$, the rôle of which is to indicate what it means for elements of A to be equal. This is implemented in type theory by stipulating that each type A comes equipped with a notion of *definitional equality on* A denoted by $=_A$ and by admitting judgements of the form:

$$(a =_A b) : A,$$

where $a, b : A$, and similarly for types in context. Moreover, for each type A and terms $a, b : A$ there is a *propositional identity type*:

$$\mathrm{Id}_A(a, b) : \mathsf{type},$$

which, in terms of the familiar Curry-Howard correspondence, is to be thought of as the proposition which states that a and b denote identical entities. Thus identity of terms is captured both via the form of judgement given by $=_A$ and via the type $\mathrm{Id}_A(a, b)$. The peculiar feature of intensional type theory (ITT) is that these two types of identity are not formally conflated by the rules of the theory. In particular, the type $\mathrm{Id}_A(a, b)$ may be inhabited without the judgement $a =_A b : A$ being derivable.

In order to see why it might be perspicuous to avoid conflating these two forms of identity consider a situation in which two functions f and g (or, if you prefer, two programs) are defined in radically different manners; but where, it so happens, they are extensionally identical. For instance:

$$\lambda x : \mathbb{N}.1$$

and,

$$\lambda x : \mathbb{N}. \underbrace{1 \times 1 \times \cdots \times 1}_{x\text{-times}}.$$

Qua ordinary set-theoretic functions these two lambda terms are identical; but *qua* algorithms they are distinct since, e.g., the former has constant time complexity whereas the latter has at least linear time complexity (and could be more complex depending on how the multiplication algorithm itself is specified).

Then, usually for pragmatic reasons, it may be convenient to formally retain their different specifications while also being able to express that they behave identically (with respect to their action on any given input). In ITT this distinction would be captured by observing that $\mathrm{Id}(f, g)$ is inhabited, but that $(f = g)$ is not a derivable judgement (or, at least, need not be derivable). Of course such situations do arise, especially in computer science, where it is convenient to retain the information about how f and g are defined (e.g., consider the case where f and g are extensionally identical but of different complexity).

This (apparent) distinction between definitional and propositional equality leads directly to interesting meta-mathematical questions. One prominent question (the question of "uniqueness of identity proofs" or UIP) is whether it is possible to show that all terms of an identity type are identical. I.e., whether given terms $a, a' : A$ and $p, q : \mathrm{Id}_A(a, a')$ one can in turn show that the type $\mathrm{Id}_{\mathrm{Id}_A(a,a')}(p, q)$ is inhabited. Such questions were taken up from both a syntactic and semantic perspective in Streicher's *Habilitationsschrift* [20] although the independence of UIP was not solved. Finally, Hofmann and Streicher [6] exhibited a counter-example to UIP using a model in the category of groupoids. One important observation leading to this result was that identity types themselves satisfy (an appropriate type theoretic form of) the axioms for groupoids. Above we considered an example (extensionally identical functions of different complexity classes) where identity types have a useful and natural interpretation. However, the observation that identity types behave like "internal groupoids" in the type theory suggests another situation in which the distinction between propositional and definitional equality is useful.

In category theory it is often convenient to conflate objects, functors and other structure which are isomorphic. Similarly, in algebraic topology there are situations where one regards two continuous maps as identical when in fact they are only homotopic to one another. These examples and those like them show how it is fruitful for two terms (maps) to be propositionally identical (i.e., naturally isomorphic or homotopic) despite the fact that they are definitionally distinct (i.e., they really are distinct maps). Identity types in intensional type theory may be regarded therefore as proving a mathematical tool for reasoning about such phenomena (cf. [6]).

Since situations where it is beneficial to systematically "ignore" certain differences between data are ubiquitous in mathematics it should come as no surprise to learn that powerful tools have been developed for investigating such matters. In particular, the concept of a *model category* was introduced by Quillen [17] to provide an axiomatic framework allowing for the development of abstract "homotopy theory" in a variety of diverse mathematical settings. As he points out, such a framework permits discussion of homotopies of simplicial sets, suspensions and loops in the setting of chain-complexes and so forth.

It is the primary goal of this thesis prospectus to show that the notion of Quillen model category provides an appropriate category theoretic setting for the semantics of intensional type theory. This is demonstrated by proving that a certain fragment of ITT can be modelled in any model category. We now turn to an overview of the prospectus and then to a discussion of the potential ramifications and applications of this research should it be carried out in full.

0.1 Overview of the prospectus

In Chapter 1 we begin by discussing the formal rules governing the behavior of identity types in intensional type theory. We then prove that these rules have no non-extensional models in the internal language of a locally cartesian closed category. This fact motivates an alternative approach to the semantics of such theories. In particular, where in a locally cartesian closed category dependent types are interpreted using slice categories we instead restrict their interpretation to those objects of the slice categories which are *fibrations*. Of course, before this semantic innovation can be made we must first introduce the appropriate mathematical background. Thus in Section 1.3 we introduce Quillen homotopy structures. This material is put to use in Section 1.4 where we prove the soundness of a restricted form of intensional type theory. Specifically, we treat in this prospectus only those aspects of intensional type theory related to the "structural" aspects of the theory and the identity types (i.e., we do not consider type formers such as dependent sums and products). As far as we are aware, the main result (Proposition 1.4.1) of Section 1.4 is entirely novel.

In Chapter 2 we turn to the question of which categories possess suitable structure to soundly interpret the restricted form of dependent type theory considered above. This material also promises to play a part in future research along these lines (particularly in proving the completeness of the theory with respect to the semantics). Specifically, in Sections 2.1 we introduce the notion of a *cocategory object* in a category. As a special case we then obtain a notion of *interval object* which permits the definition of such important notions as homotopy and fibration. These considerations are discussed in Section 2.2. One important fact about interval objects of the kind defined in this prospectus is that they endow the underlying category with the structure of a (strict) 2-category (and, indeed, an ω-category). We prove this fact in Section 2.3. Using an interval object I in a cartesian closed category with finite colimits it is possible to define the identity type for an object A to be just the map $A^I \longrightarrow A \times A$ induced by the "endpoints" $1 \rightrightarrows I$. However, in order for this to fit with the general framework it is required that this map be a fibration. In section 2.4 we provide several equivalent characterizations of what it means for this map to be a fibration. We then turn to the question of *cofibrations* in Section A.2. The definition of cocategory object is entirely standard and connections between homotopy theory and cocategory objects of various forms are well known. For example, a Hopf algebra is a form of comonoid (cf. [1]). As such it is entirely possible that the results of Chapter 2 may be already known to specialists, although we are unaware of their presence anywhere in the literature.

0.2 Future work and applications

Research relating ITT and homotopy theory or, more broadly, ITT, homotopy theory and higher dimensional category theory should ideally have applications to all three areas. We consider each of these possibilities in turn.

Possible applications to homotopy theory: One of the difficulties of homotopy theory is that it is usually a non-trivial matter to verify that a category $\mathcal{C}$ really does possess a Quillen homotopy structure. In some cases, such as for simplicial sets, it is quite involved (e.g., [7]). Moreover, aside from "brute force" the only general tool which exists for verifying that a category possess a homotopy structure is the so-called "small-object argument" (cf. [7]). It is a possibility therefore that the connection with intensional type theory may lead to new techniques for checking homotopy structure. In particular, the results of Chapter 2 suggest that there should be a general theorem about when categories with interval objects (such as the categories of (small) groupoids or simplicial sets) possess a related homotopy structure.

Possible applications to higher dimensional category theory: There is much current interest in various forms of weak higher dimensional categories (cf. [12]) and there should be some concrete connection with ITT. In particular, in ITT the usual conditions such as η type rules and the reflection rule for identity types are relaxed so that, where one demanded a canonical representative in the extensional case, one now only demands that some representative exist. This is similar to the process of relaxing identity which occurs in higher dimensional category theory. So instead of demanding, e.g., that $f \circ (g \circ h) = (f \circ g) \circ h$ one may simply stipulate that there exists a higher dimensional "cell" $f \circ (g \circ h) \Longrightarrow (f \circ g) \circ h$. In particular, there appear to be many similarities between intensional type theory and Joyal's theory of *quasi-categories* [9]; but this has not yet been investigated.

Possible applications to ITT: The most obvious way in which the tools of homotopy theory may impact ITT is by providing it with a sound and complete semantics. Such a semantics would make it possible to more easily solve meta-theoretic questions about the theory in the way that general category theoretic models of other theories do. On a more "conceptual" level, the existence of such homotopy theoretic models should provide a greater insight into some of the type forming operations such as iterated identity types $\mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_{\dots}}}(a, b)$.

More generally, it is thought that ITT should provide an "internal language" for model categories and higher dimensional categories much like λ-calculus does for cartesian closed categories. This could be an aid in proofs and calculations for homotopy and higher category theory. For instance, the fact that type checking is decidable in ITT and the close relation with a number of proof assistants suggests numerous practical applications to homotopy theory and higher dimensional category theory. Since higher dimensional category theory often involves rather complicated combinatorics it would be of great use to be able to automatically verify coherence diagrams and the like. For a nice account of the potential applications of proof assistants in mathematics see [19].

We now turn to a quick enumeration of three specific topics which should be addressed forthwith:

1. Investigate in the term model of the type theory in detail and study its connection to homotopy theory. In particular, this should permit an extension of the present semantics to the full theory with dependent products and sums.

2. One feature of ITT is that, if we wish for type checking to remain

decidable, certain η-rules must be avoided. Figuring out how to deal with this issue in the homotopy semantics is a natural next step for the project. There should also exist a connection with the notion of "weak local cartesian closure" as studied by Palmgren [16].

3. The "identity types" in several of the models arise naturally by exponentiating by an interval object I. I.e., $\mathrm{Id}_A = A^I \longrightarrow A \times A$. It is therefore natural to consider a form of intensional type theory satisfying this condition (that the identity types arise from some single type I).

0.3 Notation and terminology

Throughout we strive to maintain conventional notation sometimes relaxing the syntactic rigor of the type theory. It should be mentioned that we refer to a category as *cartesian* if it possesses all finite limits and as *cocartesian* if it possesses all finite colimits. We employ $\cdot|\cdot$ for the "judgement stroke" of type theory instead of $\cdot \vdash \cdot$. Our treatment of the type theory is very brief and it is assumed that the reader is familiar with some form of dependent type theory and their category theoretic models. In particular, we recommend [15], [20] and [5].

Chapter 1

Identity Types

It is the aim of this chapter to introduce a simple form of intensional type theory and to prove that it is sound with respect to the homotopy models introduced below. We begin by indicating the motivation for considering such models; viz. that there are no non-extensional models of the theory in the usual semantics of the internal language of a locally cartesian closed category. We begin by reviewing the interpretation of dependent type theory in a locally cartesian closed category. Since this material is standard (and is a prerequisite to understanding this prospectus) we provide only a "warm-up" discussion of judgements and refer the reader to the literature for a more detailed treatment.

1.1 The identity type in a locally cartesian closed category

As Seely [18] first observed, the internal language of a locally cartesian closed category $\mathcal{C}$ is a form of dependent type theory. In this interpretation the objects $A, B, \ldots$ of $\mathcal{C}$ determine judgements in the empty context:

$$\cdot \mid A : \mathsf{type}, \qquad \cdot \mid B : \mathsf{type}, \qquad \text{et cetera.}$$

Objects of slice categories are then types in context in the sense that an arrow $f : B \longrightarrow A$ is a judgement:

$$A \mid B : \mathsf{type},$$

which we will sometimes write as either $A \mid_f B : \mathsf{type}$ or $A \mid f : \mathsf{type}$ to emphasize the rôle of the map f. Using the fact that $(\mathcal{C}/A)/f \cong \mathcal{C}/\operatorname{dom}(f)$ whenever A is an object of $\mathcal{C}$ and f is an object of $\mathcal{C}/A$ we are able to interpret more complicated contexts. For example, the judgement:

$$x_0 : A_0, \ldots, x_n : A_n \mid g : \mathsf{type}$$

asserts that there exists a "chain" of arrows:

$$\mathrm{dom}(g) \xrightarrow{g} A_n \xrightarrow{f_n} \dots \xrightarrow{f_1} A_0$$

in $\mathcal{C}$. Where Γ is the context $(x_0 : A_0, \dots, x_n : A_n)$ as above we write $\Gamma : A_n \longrightarrow A_0$ for the composite $f_1 \circ \dots \circ f_n$. As such, the claim $\Gamma \mid_g B : \mathsf{type}$ means that there exists an object g of $\mathcal{C}/\,\mathrm{dom}(\Gamma)$ with $\mathrm{dom}(g) = B$.

A judgement $\cdot \mid a : A$ may be regarded as stating that there exists a global section $a : 1 \longrightarrow A$ of A in $\mathcal{C}$. A judgement in context $\Gamma \mid b : g$ or $\Gamma \mid_g b : B$ indicates the existence a global section of the object g in the slice category $\mathcal{C}/\,\mathrm{dom}(\Gamma)$. That is, b is a map $\mathrm{dom}(\Gamma) \longrightarrow \mathrm{dom}(f)$ in $\mathcal{C}$ such that the following triangle:

$$\begin{tikzcd} A \arrow[rr, "b"] \arrow[dr, "1_A"'] & & B \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

commutes, where $B = \mathrm{dom}(g)$ and $A = \mathrm{dom}(\Gamma)$.

We can now investigate the (propositional) identity types Id in a locally cartesian closed category $\mathcal{C}$. First, the formation rule for Id is as follows:

$$\frac{\Gamma \mid A : \mathsf{type}}{\Gamma, x : A, y : A \mid \mathrm{Id}_A(x, y) : \mathsf{type}.}\ \text{Id-Form}$$

In order to simplify notation let us work in the empty context $\Gamma = (\cdot)$. Suppose A is an object of $\mathcal{C}$. Then the Id-Form rule indicates that Id_A should be an object of $\mathcal{C}/(A \times A)$ since we have applied the weakening rule once in order to form the context $(\Gamma, x : A, y : A)$. I.e., our candidate for Id_A should be of the form:

$$\mathrm{Id}_A \xrightarrow{i} A \times A.$$

Next, the introduction rule provides another clue about how Id_A should behave. Informally, the introduction rule states that for any $a : A$ there is always a "witness" for the proposition that a is identical to itself:

$$\frac{\Gamma \mid A : \mathsf{type}}{\Gamma, x : A \mid r_A(x) : \mathrm{Id}_A(x, x).}\ \text{Id-Intro}$$

The term $r_A(x)$ is referred to as the *reflexivity term*. Working again in the empty context, this rule asserts the existence of a global section of $\mathrm{Id}_A(x, x)$

in $\mathcal{C}/A$, where $\mathrm{Id}_A(x,x)$ is the result of pulling i back along the diagonal $\Delta : A \longrightarrow A \times A$. Equivalently, there exists a section $r_A : A \longrightarrow \mathrm{Id}_A$ as indicated in the following diagram:

$$\begin{tikzcd} A \arrow[rr, "r_A"] \arrow[dr, "\Delta"'] & & \mathrm{Id}_A \arrow[dl, "i"] \\ & A \times A. & \end{tikzcd} \tag{1.1}$$

Finally, the elimination rule can be stated as follows:

$$\frac{\begin{array}{c}\Gamma, x : A, y : A, z : \mathrm{Id}_A(x,y) \mid B[x,y,z] : \mathsf{type} \\ \Gamma, u : A \mid b(u) : B[u,u,r_A(u)]\end{array}}{\Gamma, x : A, y : A, z : \mathrm{Id}_A(x,y) \mid J_{A,B}(b,x,y,z) : B[x,y,z].}\ \text{Id-Elim} \tag{1.2}$$

Informally, this rather complicated looking rule states that if B is a type which varies over Id_A and there is a "generic proof" b that B holds for all (identical terms and their) reflexivity terms, then, whenever there exists a proof z that two terms x and y of type A are propositionally identical, there is also a proof J that B is true of x, y and z. Briefly, if B holds for (x,x) and $r(x)$, then it also holds for (x,y) and z, if z witnesses the propositional identity of x and y — and this inference is itself witnessed.

In terms of the categorical semantics this rule means that, given $g : B \longrightarrow \mathrm{Id}_A$, any section:

$$\begin{tikzcd} A \arrow[rr, "b"] \arrow[dr, "r_A"'] & & B \arrow[dl, "g"] \\ & \mathrm{Id}_A, & \end{tikzcd}$$

extends to a global section $J : \mathrm{Id}_A \longrightarrow B$:

$$\begin{tikzcd} \mathrm{Id}_A \arrow[rr, "J"] \arrow[dr, "1_{\mathrm{Id}_A}"'] & & B \arrow[dl, "g"] \\ & \mathrm{Id}_A. & \end{tikzcd}$$

Finally, the conversion rule for the J term is as follows:

$$\frac{}{\Gamma \mid J_{A,B}(b,a,a,r_A(a)) = b(a) : B(a,a,r_A(a)).}\ \text{Id-Conv} \tag{1.3}$$

The elimination and conversion rule then fit together nicely as stating that for any g and b as above there exists a map $J : \mathrm{Id}_A \longrightarrow B$ making the following pyramid commute:

$$\begin{tikzcd}
 & B \arrow[dd, "g"] & \\
A \arrow[ur, "b"] \arrow[rr] \arrow[dr, "r_A"'] & & \mathrm{Id}_A, \arrow[ul, dotted, "J"'] \arrow[dl, "1_{\mathrm{Id}_A}"] \\
 & \mathrm{Id}_A &
\end{tikzcd} \tag{1.4}$$

where the arrow $A \longrightarrow \mathrm{Id}_A$ along the back face is r_A.

1.2 Locally cartesian closed categories are extensional

As indicated in the introduction, the distinctive feature of ITT is that the notion of propositional equality codified by the rules governing the identity types Id is now allowed to be distinct from the notion of *definitional (judgemental) equality* given by judgements of the form $\Gamma|a = b : A$. In particular, the distinction between intensional and extensional identity is captured by the fact that the following *identity reflection rule* is not in general valid in ITT:

$$\frac{\Gamma \mid p : \mathrm{Id}_A(a_0, a_1)}{\Gamma \mid a_0 = a_1 : A.}\ \text{Id-Reflection}$$

Note that the converse holds as a consequence of the introduction rule for Id_A. By saying that a model (or semantics) for ITT is *extensional* we mean that it validates Id-Reflection.

The following result seems to be part of the "folk-lore" of the subject (although we know of no mention of it in the literature). We learned of it from Steve Awodey, to whom the following proof is due.

Proposition 1.2.1 (All LCC "models" are extensional) *Under the standard interpretation given above, every locally cartesian closed category $\mathcal{C}$ is extensional.*

Proof We prove that any candidate identity type Id_A which satisfies the introduction and elimination rules for propositional identity is isomorphic to the diagonal $\Delta : A \longrightarrow A \times A$. It suffices to verify this for the case where A is a type in the empty context.

By assumption we have $i : \mathrm{Id}_A \longrightarrow A \times A$ and a map $r_A : A \longrightarrow \mathrm{Id}_A$ such that (1.1) commutes. Moreover, r_A may itself then be regarded as a type over Id_A:

$$x : A, y : A, z : \mathrm{Id}_A(x, y) \mid r_A : \mathsf{type}.$$

I.e., we may apply the elimination rule to r_A. Therefore there exists a map $J : \mathrm{Id}_A \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Id}_A \arrow[rr, "J"] \arrow[dr, "1_{\mathrm{Id}_A}"'] & & A \arrow[dl, "r_A"] \\
& \mathrm{Id}_A. &
\end{tikzcd}$$

Therefore $\mathrm{Id}_A \cong A$. □

Our leading idea for developing a category theoretic semantics for ITT in which there exist non-extensional models is to place additional constraints on which objects and arrows are allowed to play the rôle of (dependent) types. In particular, the types should be restricted in such a way as to exclude (in general) the map $r_A : A \longrightarrow I_A$. Before we can introduce this new interpretation it is first necessary to take a detour through Quillen's abstract homotopy theory.

1.3 Quillen homotopy structures

We recall the definition of *Quillen homotopy structures* (synonymously, *Quillen model categories*). References include [17], [4], [7] and the forthcoming [10]. Our presentation follows that of the latter most closely.

Definition 1.3.1 A *Quillen homotopy structure* on a cartesian and cocartesian category $\mathcal{C}$ consists of three classes of maps $(\mathfrak{F}, \mathfrak{C}, \mathfrak{W})$ referred to as *fibrations*, *cofibrations* and *weak equivalences*, respectively, which satisfy the following axioms.

(Q0) Both $\mathfrak{F}$, $\mathfrak{C}$ and $\mathfrak{W}$ are stable under composition and contain all identity maps.

(Q1) Given maps f and g in $\mathcal{C}$ such that the composite $g \circ f$ is defined, if any two of f, g or $g \circ f$ is in $\mathfrak{W}$, then so is the third. This axiom is called the *saturation* or *three-for-two* axiom.

(Q2) Given a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[r, "j"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "u"'] & Y' \arrow[r, "v"'] & Y, \end{tikzcd}$$

such that $j \circ i = 1_X$ and $v \circ u = 1_Y$, then if f' is in $\mathfrak{F}$, $\mathfrak{C}$ or $\mathfrak{W}$, then so is f. This is called the *retracts* axiom.

(Q3) Let a commutative square be given as follows:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r] & D \end{tikzcd}$$

with p a fibration and i a cofibration. Then if either p or i is also a weak-equivalence, there exists a *diagonal filler* $h : B \longrightarrow C$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r] \arrow[ur, dotted, "h"] & D \end{tikzcd}$$

making both triangles commute. This is called the *lifting* axiom.

(Q4) Any map $f : A \longrightarrow B$ in $\mathcal{C}$ has two factorizations of the form:

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

such that in both factorizations p is a fibration and i is a cofibration and in one factorization p is a weak-equivalence and in the other i is a weak equivalence. This is called the *factorization* axiom.

We say that a map which is in $(\mathfrak{F} \cap \mathfrak{W})$ is an *acyclic fibration* and a map which is in $(\mathfrak{C} \cap \mathfrak{W})$ is an *acyclic cofibration.* The reader should be aware however that the terms “trivial fibration” and “trivial cofibration” are used frequently in the literature.

Example 1.3.2 The following are examples of model categories:

1. Given any cartesian and cocartesian category $\mathcal{C}$ let one of $\mathfrak{F}, \mathfrak{C}, \mathfrak{W}$ be all isomorphisms of $\mathcal{C}$ and the other two be all maps.

2. The category **Gpds** of small groupoids and functors between them has a homotopy structure where the fibrations are the *Grothendieck fibrations* (cf. Definition 2.5.5), cofibrations are functors which are injective on objects, and weak equivalences are categorical equivalences. This is one of the motivating examples and will be considered in detail below in Chapter 2.

3. The category $\mathbf{Top}_c$ of compactly generated spaces is a model category where the fibrations are *Serre fibrations*, the weak equivalences are the topological weak equivalences, and the cofibrations are those maps which have the left-lifting property (see below) with respect to the maps in $\mathfrak{F} \cap \mathfrak{W}$.

4. An alternative homotopy structure on $\mathbf{Top}_c$ is given by the *Hurewicz fibrations*, *Hurewicz cofibrations* and homotopy equivalences.

5. The category $\mathbf{S}$ of simplicial sets is a model category where the fibrations are *Kan fibrations*, the cofibrations are monomorphisms, and the weak equivalences are geometric homotopy equivalences.

One of the tricky points about Quillen's homotopy theory is that proving a category has a homotopy structure is often highly non-trival. As checking these examples would take us too far afield we instead refer the reader to the literature: [7], [4], [17] and [10].

Definition 1.3.3 A map $f : A \longrightarrow B$ has the *left-lifting property* (LLP) with respect to another map $g : C \longrightarrow D$ if for any commutative square:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd}$$

there exists a diagonal filler $l : B \longrightarrow C$ such that $l \circ f = h$ and $g \circ l = k$. In this situation we also say that g has the *right-lifting property* (RLP) with respect to f and write $f \perp g$.

Similarly, given two collections of maps $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{C}$ we say that $\mathcal{G}$ has the LLP with respect to $\mathcal{H}$ and $\mathcal{H}$ has the RLP with respect to $\mathcal{G}$ if and only if, for any g in $\mathcal{G}$ and h in $\mathcal{H}$, $g \perp h$. In this case we write $\mathcal{G} \perp \mathcal{H}$.

Given any collection $\mathcal{G}$ of maps in $\mathcal{C}$ we write ${}^{\perp}\mathcal{G}$ for the collection of maps which have the LLP with respect to $\mathcal{G}$ and $\mathcal{G}^{\perp}$ for the collection of maps with the RLP with respect to $\mathcal{G}$.

In terms of Definition 1.3.3 axiom **(Q3)** states that the following hold:

$$\begin{aligned} &(\mathfrak{C} \cap \mathfrak{W}) \perp \mathfrak{F}, \qquad \text{and} \\ &\mathfrak{C} \perp (\mathfrak{F} \cap \mathfrak{W}). \end{aligned}$$

The following lemma strengthens these relations by showing that the fibrations are exactly those maps with the RLP with respect to acyclic cofibrations and the cofibrations are exactly those maps with the LLP with respect to acyclic fibrations.

Lemma 1.3.4 (The Retracts Argument) *If $\mathcal{C}$ is a category with a Quillen homotopy structure, then:*

$$\mathfrak{F} = (\mathfrak{C} \cap \mathfrak{W})^{\perp}, \tag{1.5}$$

$$\mathfrak{C} = {}^{\perp}(\mathfrak{F} \cap \mathfrak{W}), \tag{1.6}$$

$$\mathfrak{C}^{\perp} = \mathfrak{F} \cap \mathfrak{W}, \text{ and} \tag{1.7}$$

$${}^{\perp}\mathfrak{F} = \mathfrak{C} \cap \mathfrak{W}. \tag{1.8}$$

Proof All four arguments are essentially the same. As such, we prove only (1.5). Suppose $f : A \longrightarrow B$ has the RLP with respect to acyclic cofibrations. By **(Q4)** we may factor f as $p \circ i$ where $i : A \longrightarrow C$ is an acyclic cofibration and $p : C \longrightarrow B$ is a fibration. As such there exists a diagonal filler $h : C \longrightarrow A$ as indicated in the following diagram:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "i"'] & A \arrow[d, "f"] \\ C \arrow[r, "p"'] \arrow[ur, dotted, "h"] & B. \end{tikzcd}$$

But then the following is a retract diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[r, "h"] \arrow[d, "p"] & A \arrow[d, "f"] \\ B \arrow[r, "1_B"'] & B \arrow[r, "1_B"'] & B, \end{tikzcd}$$

so f is a fibration by **(Q2)**. $\square$

Lemma 1.3.5 *If $\mathcal{C}$ is a category with Quillen homotopy structure, then both $\mathfrak{F}$ and $\mathfrak{F} \cap \mathfrak{W}$ are stable under base change (pullback). Similarly, both $\mathfrak{C}$ and $\mathfrak{C} \cap \mathfrak{W}$ are stable under cobase change (pushout).*

PROOF By Lemma 1.3.4. □

Definition 1.3.6 Given an object A of a model category $\mathcal{C}$ a *cylinder object for A* is an object $\mathrm{Cyl}(A)$ together with a factorization of the codiagonal:

$$\begin{tikzcd} A + A \arrow[rr, "\varepsilon"] \arrow[dr, "\nabla"'] & & \mathrm{Cyl}(A) \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

such that f is a weak equivalence. $\mathrm{Cyl}(A)$ is a *good cylinder object for A* if ε is a cofibration and it is a *very good cylinder object for A* if it is good and f is a fibration.

Dually, a *path object for A* is an object $\mathrm{Path}(A)$ together with a factorization of the diagonal:

$$\begin{tikzcd} A \arrow[rr, "r"] \arrow[dr, "\Delta"'] & & \mathrm{Path}(A) \arrow[dl, "\iota"] \\ & A \times A & \end{tikzcd}$$

such that r is a weak equivalence. $\mathrm{Path}(A)$ is *good* if ι is a fibration and *very good* if it is good and r is a cofibration.

Axiom **(Q4)** implies that every object has a very good cylinder object and a very good path object.

Definition 1.3.7 An object A of a model category $\mathcal{C}$ is *fibrant* if the canonical map $A \longrightarrow 1$ is a fibration. A is *cofibrant* if the canonical map $0 \longrightarrow A$ is a cofibration.

1.4 The interpretation

Assuming that the ambient category $\mathcal{C}$ is a model category we may interpret dependent type theory now by using just the fibrations over an object A, instead of arbitrary objects in $\mathcal{C}/A$, to interpret the dependent types over A.

Specifically, a judgement $\cdot \mid A : \mathsf{type}$ in the empty context indicates that A is a fibrant object of $\mathcal{C}$. A judgement of the form $x : A|_f B : \mathsf{type}$

then indicates that $f : B \longrightarrow A$ is a fibration. In particular, a context $(x_0 : A_0, \ldots, x_n : A_n)$ is well-formed if and only if $\mathrm{dom}(f_{m-1}) = \mathrm{cod}(m)$ for each $1 \leq m \leq n$ and each map f_i is a fibration:

$$A_n \xrightarrow{f_n} A_n - 1 \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} A_0.$$

Note that fibrations in $\mathcal{C}/A$ are those maps which are already fibrations in $\mathcal{C}$ since the model category structure is preserved by slicing.

Choose for each object A of $\mathcal{C}$ and each f of $\mathcal{C}/A$ very good path objects and denote them by Id_A and Id_f, respectively. As the notation indicates these objects, together with the associated maps r and ι, will serve to interpret the identity types.

Proposition 1.4.1 (Warren) *Let $\mathcal{C}$ be a model category equipped with a choice of very good path object for each object A of $\mathcal{C}$ and object f of $\mathcal{C}/A$. Then the rules* Id-Form*,* Id-Intro*,* Id-Elim *and* Id-Conv *are all valid in $\mathcal{C}$.*

Proof The formation rule and introduction rules are valid by definition of the interpretation of Id_A as a very good path object for A. I.e., the formation rule is valid because ι is a fibration and the introduction rule is valid since $\iota \circ r$ is the diagonal. To see that the elimination and conversion rules hold it suffices to work in the empty context since homotopy structure is stable under slicing. Assume that the following judgements are derivable:

$$x : A, y : A, z : \mathrm{Id}_A(x, y) \mid_g B : \mathsf{type} \quad \text{and} \quad x : A \mid b : B(x, x, r(x)).$$

We then require $J : \mathrm{Id}_A \longrightarrow B$ such that $g \circ J = 1_{\mathrm{Id}_A}$.

By definition of the interpretation, $g : B \longrightarrow \mathrm{Id}_A$ is a fibration and there exists a map $b : A \longrightarrow B$ with $g \circ b = r$. Putting this together we have that the following square commutes:

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "r"'] & B \arrow[d, "g"] \\ \mathrm{Id}_A \arrow[r, "1_{\mathrm{Id}_A}"'] & \mathrm{Id}_A. \end{tikzcd}$$

But g is a fibration and r is, by definition, an acyclic cofibration. Therefore there exists a diagonal filler $J : \mathrm{Id}_A \longrightarrow B$:

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "r"'] & B \arrow[d, "g"] \\ \mathrm{Id}_A \arrow[r, "1_{\mathrm{Id}_A}"'] \arrow[ur, dotted, "J"] & \mathrm{Id}_A. \end{tikzcd}$$

Commutativity of the bottom triangle is precisely the conclusion of the elimination rule:

$$\Gamma, x : A, y : A, z : \mathrm{Id}_A(x, y) \mid J_{A,B}(b, x, y, z) : B[x, y, z],$$

and commutativity of the top triangle is the conversion rule:

$$\Gamma \mid J_{A,B}(b, a, a, r_A(a)) = b(a) : B(a, a, r_A(a)). \qquad \square$$

Chapter 2

Cocategory Objects

In this chapter we consider cartesian closed categories with a "interval object" and how this might be used to prove that various categories possess homotopy structure (and thus model ITT). Although this way of approaching the model structure of categories is yet incomplete the axioms which we introduce do provide a fair amount of homotopy structure. In particular, Proposition 2.4.2 should be mentioned as it provides an interesting characterization of the circumstances under which the "identity type" (induced by the unit interval) will actually be a type (i.e., a fibration). We end by showing that the category **Gpds** of small groupoids satisfies these axioms and that the resulting homotopy structure agrees with the usual one for this category.

2.1 Cocategory objects

The notion of *cocategory object* or *internal cocategory* is dual to that of a category object or internal category (cf. [13], [8] or [3]). Rather than rehearse the latter we state the definition of cocategory object directly.

Definition 2.1.1 A *cocategory object* $\mathbb{C}$ in a category $\mathcal{C}$ with pushouts consists of the following data.

Objects: C_0 (*object of coobjects*), C_1 (*object of coarrows*) and C_2 (*object of cocomposable coarrows*).

Arrows: $\bot, \top : C_0 \rightrightarrows C_1$ (*bottom* and *top*), $i : C_1 \longrightarrow C_0$ (*coidentities*), $\downarrow, \uparrow : C_1 \rightrightarrows C_2$ (*initial segment* and *final segment*), and $* : C_1 \longrightarrow C_2$ (*cocomposition*).

Satisfying the following list of requirements.

1. The following square is a pushout:

$$\begin{tikzcd} C_0 \arrow[r, "\perp"] \arrow[d, "\top"'] & C_1 \arrow[d, "\uparrow"] \\ C_1 \arrow[r, "\downarrow"'] & C_2. \end{tikzcd} \tag{2.1}$$

2. The following diagram commutes:

$$\begin{tikzcd} C_0 \arrow[r, "\perp"] \arrow[dr, equal] & C_1 \arrow[d, "i"] & C_0 \arrow[l, "\top"'] \arrow[dl, equal] \\ & C_0. & \end{tikzcd} \tag{2.2}$$

3. The following diagrams commute:

$$\begin{tikzcd} C_0 \arrow[r, "\perp"] \arrow[d, "\perp"'] & C_1 \arrow[d, "*"] \\ C_1 \arrow[r, "\downarrow"'] & C_2, \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_0 \arrow[r, "\top"] \arrow[d, "\top"'] & C_1 \arrow[d, "*"] \\ C_1 \arrow[r, "\uparrow"'] & C_2. \end{tikzcd} \tag{2.3}$$

4. Notice that since (2.1) is a pushout there exist, by (2.2), canonical maps $i_0, i_1 : C_2 \rightrightarrows C_1$ such that:

$$\begin{aligned} i_0 \circ \downarrow &= \perp \circ\, i, \\ i_0 \circ \uparrow &= 1_{C_1}, \\ i_1 \circ \uparrow &= \top \circ\, i, \text{ and} \\ i_1 \circ \downarrow &= 1_{C_1}. \end{aligned}$$

We stipulate, moreover, that the following *counit* diagram commutes:

$$\begin{tikzcd} & C_1 \arrow[dl, equal] \arrow[d, dashed, "*"] \arrow[dr, equal] & \\ C_1 & C_2 \arrow[l, "i_0"] \arrow[r, "i_1"'] & C_1. \end{tikzcd} \tag{2.4}$$

5. Finally, let the object C_3 (*the object of cocomposable triples*) be defined as the following pushout:

$$\begin{tikzcd} C_1 \arrow[r, "\uparrow"] \arrow[d, "\downarrow"'] & C_2 \arrow[d, "q_1"] \\ C_2 \arrow[r, "q_0"'] & C_3, \end{tikzcd}$$

and observe that (by the dual of the "two-pullbacks" lemma) C_3 may be alternatively described as the following pushout:

$$\begin{tikzcd} C_0 \arrow[r, "\downarrow\circ\perp"] \arrow[d, "\top"'] & C_2 \arrow[d, "q_0"] \\ C_1 \arrow[r, "r_0"'] & C_3, \end{tikzcd}$$

where $r_0 := q_1 \circ \downarrow$ or as the pushout of $\uparrow \circ \top$ along $\perp$:

$$\begin{tikzcd} C_0 \arrow[r, "\uparrow\circ\top"] \arrow[d, "\perp"'] & C_2 \arrow[d, "q_1"] \\ C_1 \arrow[r, "r_1"'] & C_3, \end{tikzcd}$$

where $r_1 := q_0 \circ \uparrow$.

There exist then canonical maps $t_0, t_1 : C_2 \rightrightarrows C_3$ such that:

$$\begin{aligned} t_0 \circ \downarrow &= r_0, \\ t_0 \circ \uparrow &= q_0 \circ *, \\ t_1 \circ \downarrow &= q_1 \circ *, \text{ and} \\ t_1 \circ \uparrow &= r_1. \end{aligned}$$

The *coassociative law* then states that the following diagram commutes:

$$\begin{tikzcd} C_1 \arrow[r, "*"] \arrow[d, "*"'] & C_2 \arrow[d, "t_0"] \\ C_2 \arrow[r, "t_1"'] & C_3. \end{tikzcd} \tag{2.5}$$

Several comments about this definition are in order. Although some of the nomenclature employed is at this point unfamiliar it is justified below when we explain our intended interpretation (also, it allows us to avoid such repugnant locutions as "cocodomain"). In particular, $\perp$ is the dual of a domain map, $\top$ is the dual of a codomain map, and $\downharpoonleft$ and $\upharpoonleft$ are dual to the first and second projections, respectively.

Example 2.1.2 The following are examples of cocategory objects.

1. Given an object A of a category $\mathcal{C}$ one always has the "trivial cocategory" on A given by setting $C_i := A$ for $i = 0, 1, 2$ and $\perp = \top = \downharpoonleft = \upharpoonleft = * = 1_A$.

2. If $\mathbb{D}$ is a cocategory in a category $\mathcal{C}$ with products which preserve pushouts, then $A \times \mathbb{D}$ is also a cocategory for any object A of $\mathcal{C}$.

3. In **Sets** the two element set 2 is a cocategory (where $C_0 := 1$, $C_1 := 2$ and $C_3 := 3$ with $\perp, \top, \downharpoonleft, \upharpoonleft$ and $*$ defined as one would expect).

4. In Section 2.2 below we introduce a cocategory object in the category **Gpds** of small groupoids.

Proposition 2.1.3 *If $\mathcal{C}$ is a cocartesian ccc with a cocategory object $\mathbb{C}$, then, for any object D of $\mathcal{C}$, the slice category $\mathcal{C}/D$ also possesses a cocategory object $\mathbb{C}_D$. Moreover, if $f : B \longrightarrow D$ is an arrow in $\mathcal{C}$, then $\Delta_f : \mathcal{C}/D \longrightarrow \mathcal{C}/B$ preserves the cocategory structure.*

PROOF The cocategory object $\mathbb{C}_D$ is given by forming the product with D. I.e., the object of coobjects is simply the projection $\Delta_D(C_0)$ given by $D \times C_0 \longrightarrow D$. Since $\mathcal{C}$ is a ccc all of the relevant pushout diagrams are preserved. Since all of the other data is equational it is clear that this is a cocategory object in $\mathcal{C}/D$. It is also clear that this structure is preserved by pullback. $\square$

2.2 Homotopy in a CCC with interval object

We will be concerned with cocategory objects with certain additional properties.

Definition 2.2.1 A cocategory object $\mathbb{C}$ in a category $\mathcal{C}$ is *pointed* if the object C_0 of coobjects is the terminal object of $\mathcal{C}$. $\mathbb{C}$ is *reversible* if there exists a map $\rho : C_1 \longrightarrow C_1$ (*the reversal map*) such that $\rho(\perp) = \top$ and $\rho(\top) = \perp$. Finally, $\mathbb{C}$ is a *strict interval object* if it is both pointed and reversible. When $\mathbb{C}$ is a strict interval object we will write I instead of C_1 and I_2 instead of C_2. We will also omit the word “strict” when referring to interval objects.

The reader should see Appendix A.1 for a geometric illustration of the definition of interval object.

Example 2.2.2 In the category **Gpds** of small groupoids the connected two element groupoid $\mathbf{I}$ is an interval object:

$$\begin{tikzcd} \top \arrow[d, bend right, "d"'] \\ \bot, \arrow[u, bend right, "u"'] \end{tikzcd}$$

with $\bot, \top : \mathbf{1} \rightrightarrows \mathbf{I}$ the obvious functors. $\mathbf{I}_2$ is then the result of gluing $\mathbf{I}$ to itself along the top and bottom:

$$\begin{tikzcd} \top \arrow[d, bend right, "d_\uparrow"'] \\ \mu \arrow[u, bend right, "u_\uparrow"'] \arrow[d, bend right, "d_\downarrow"'] \\ \bot. \arrow[u, bend right, "u_\downarrow"'] \end{tikzcd}$$

Cocomposition $* : \mathbf{I} \longrightarrow \mathbf{I}_2$ is the unique functor given by $*(\bot) := \bot$ and $*(\top) := \top$, and the initial and final segment functors are defined in the evident way. Finally, $\rho : \mathbf{I} \longrightarrow \mathbf{I}$ is defined by $\rho(\bot) := \top$ and $\rho(\top) := \bot$.

Interval objects of this kind are useful in so far as they provide us with a way of defining (abstractly) such notions as *homotopy*, *fibration* and *weak equivalence*.

Definition 2.2.3 Let $\mathcal{C}$ be a cartesian closed category which is also cocartesian an possesses an interval object I. A *homotopy* $\eta : f \Longrightarrow g$ between two maps $f, g : A \rightrightarrows B$ in $\mathcal{C}$ is a map $\eta : A \times I \longrightarrow B$ such that the following triangles commute:

$$\begin{tikzcd} A \arrow[r, "A_0"] \arrow[dr, "f"'] & A \times I \arrow[d, "\eta"] & A \arrow[l, "A_1"'] \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

where $A_0 := \langle 1_A, \bot \circ !_A \rangle$ and $A_1 := \langle 1_A, \top \circ !_A \rangle$. One can think of $A \times I$ as an abstract "cylinder" with A_0 and A_1 as the inclusions at the two ends.

A map $p : E \longrightarrow B$ in $\mathcal{C}$ is a *(Hurewicz) fibration* if for any object A, and maps $f : A \longrightarrow E$ and $h : A \times I \longrightarrow B$ there exists a diagonal filler:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "A_0"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "h"'] \arrow[ur, dotted] & B. \end{tikzcd}$$

I.e., $\mathfrak{I} \perp p$ where $\mathfrak{I}$ is the collection of all maps of the form A_0 for A an object of $\mathcal{C}$.

A map $f : A \longrightarrow B$ is a *weak equivalence* if and only if there exists a map $f' : B \longrightarrow A$ and homotopies $f \circ f' \Longrightarrow 1_B$ and $f' \circ f \Longrightarrow 1_A$.

Remark 2.2.4 Notice that given a homotopy $\eta : f \Longrightarrow g$ between maps $f, g : A \rightrightarrows B$ we obtain a homotopy $\bar{\eta} : g \Longrightarrow f$ by composing with the reverse map:

$$\bar{\eta} \quad := \quad \eta \circ (1_A \times \rho).$$

For the remainder of this section we assume that $\mathcal{C}$ is a category satisfying the assumptions of Definition 2.2.3.

Lemma 2.2.5 *The collection $\mathfrak{F}$ of fibrations in $\mathcal{C}$ has the following properties:*

1. *$\mathfrak{F}$ is stable under composition. I.e., if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are in $\mathfrak{F}$, then so is the composite $g \circ f$. Moreover, all isomorphisms are contained in $\mathfrak{F}$.*

2. *(Base change) $\mathfrak{F}$ is stable under pullback along arbitrary maps. I.e., in any pullback square:*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B' \arrow[r] & B \end{tikzcd}$$

if f is in $\mathfrak{F}$, then so is f'.

3. *Every object of $\mathcal{C}$ is fibrant.*

4. Product projections are fibrations.

PROOF Stability under composition and base change are straightforward. It is trivial to see that every object is fibrant. Finally, (4) is by (2) and (3). □

As Lemma 2.2.5 suggests the fibrations in this abstract setting already have useful properties. However, before developing them further it will be instructive to first investigate (some of) the higher dimensional structure on $\mathcal{C}$ induced by the interval object I.

2.3 2-categorical structure induced by I

Let $\mathcal{C}$ be a cocartesian ccc with interval I. Because I is, by definition, a cocategory it follows that for any objects A and B of $\mathcal{C}$:

$$\mathrm{Hom}(A \times I, B) \;\cong\; \mathrm{Hom}(A, B^I),$$

is a category. This therefore trivially induces a 2-categorical structure on $\mathcal{C}$. In this section we will elaborate on this point by providing a detailed verification of the 2-categorical structure. This process should help familiarize the reader with interval objects and the induced 2-categorical structure.

Lemma 2.3.1 (Homotopies compose) *Given maps f, g and h with domain A and codomain B, if $\eta : f \Longrightarrow g$ and $\gamma : g \Longrightarrow h$, then there exists a homotopy $\delta : f \Longrightarrow h$.*

PROOF First observe that since $\mathcal{C}$ is cartesian closed the following square is a pushout:

$$\begin{tikzcd} A \times 1 \arrow[r, "1_A \times \top"] \arrow[d, "1_A \times \bot"'] & A \times I \arrow[d, "1_A \times \downarrow"] \\ A \times I \arrow[r, "1_A \times \uparrow"'] & A \times I_2. \end{tikzcd}$$

Then, since $\eta \circ A_1 = \gamma \circ A_0$, there exists a canonical map $\delta : A \times I_2 \longrightarrow B$ such that:

$$\begin{aligned} \delta \circ (1_A \times \uparrow) &= \gamma, \text{ and} \\ \delta \circ (1_A \times \downarrow) &= \eta. \end{aligned}$$

Recalling the third clause from the definition of cocategory object, it is easily verified that $\delta \circ (1_A \times *)$ is the required homotopy. □

Henceforth, given homotopies η and γ as in the statement of Lemma 2.3.1, we write $(\gamma \cdot \eta) : f \Longrightarrow h$ for the homotopy $\delta \circ (1_A \times *)$ constructed in the proof and refer to this as the *vertical composition of* η *and* γ. It is also convenient to introduce notation for the "mediating map" δ. As such, we write $c[\gamma, \eta]$ instead of δ and observe that $(\gamma \cdot \eta) = c[\gamma, \eta] \circ (1_A \times *)$. I.e., $c[\gamma, \eta]$ is "data" for the composition $(\gamma \cdot \eta)$ prior to being "fused" or "merged" by precomposition with $(1_A \times *)$.

Remark 2.3.2 Given homotopies $\alpha, \beta : A \times I \longrightarrow B$ for which the vertical composite $(\beta \cdot \alpha)$ exists and any map $g : D \longrightarrow A$:

$$c[\beta, \alpha] \circ (g \times 1_{I_2}) \quad = \quad c[\beta \circ (g \times 1_I), \alpha \circ (g \times 1_I)].$$

Assume given a "2-dimensional" diagram involving objects, arrows and homotopies in $\mathcal{C}$:

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\eta\Downarrow"] & D \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\gamma\Downarrow"] & E. \end{tikzcd}$$

Then we define the *horizontal composition* $\gamma \star \eta$ to be the homotopy $C \times I \longrightarrow E$ given by the composite:

$$C \times I \xrightarrow{1_C \times \Delta} C \times I \times I \xrightarrow{\eta \times 1_I} D \times I \xrightarrow{\gamma} E,$$

where $\Delta : I \rightarrowtail I \times I$ is the diagonal. This is clearly a homotopy $h \circ f \Longrightarrow k \circ g$.

Lemma 2.3.3 (Interchange) *Let the following diagram be given:*

$$\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & D \arrow[r, bend left=60, "k"] \arrow[r, "l"] \arrow[r, bend right=60, "m"'] \arrow[r, phantom, shift left=3, "\gamma\Downarrow"] \arrow[r, phantom, shift right=3, "\delta\Downarrow"] & E, \end{tikzcd}$$

Then the Interchange Law *holds:*

$$(\delta \star \beta) \cdot (\gamma \star \alpha) \quad = \quad (\delta \cdot \gamma) \star (\beta \cdot \alpha).$$

PROOF Observe that:

$$(\delta \cdot \gamma) \star (\beta \cdot \alpha) \quad = \quad c[\gamma, \delta] \circ \big(c[\beta, \alpha] \times 1_I\big) \circ (1_C \times \Delta) \circ (1_C \times *).$$

Therefore it suffices to show that:

$$c[\gamma,\delta] \circ \big(c[\beta,\alpha] \times 1_I\big) \circ (1_C \times \Delta) \quad = \quad c[\delta \star \beta, \gamma \star \alpha],$$

which is a straightforward application of the definitions. □

Proposition 2.3.4 *Let $\mathcal{C}$ be a cartesian closed category which is cocartesian and possesses an interval object I. Then $\mathcal{C}$ is a 2-category (cf. Appendix A.3) with the same objects and arrows, and with 2-cells given by homotopies.*

Proof We have already defined vertical and horizontal composition of 2-cells. By Lemma 2.3.3 the interchange law holds. All that remains is to verify that the compositions are associative and that identities exist.

Given an arrow $f : A \longrightarrow B$ the identity homotopy $1_f : f \Longrightarrow f$ is defined to be $f \circ \pi_A : A \times I \longrightarrow B$. Assume given a homotopy $\gamma : f \Longrightarrow g$ in order to show that $(\gamma \cdot 1_f) = \gamma$. Observe that:

$$\gamma \quad = \quad \gamma \circ (1_A \times i_0) \circ (1_A \times *),$$

by (2.4) from Definition 2.1.1. Therefore it suffices to prove that:

$$\gamma \circ (1_A \times i_0) \quad = \quad c[\gamma, 1_f],$$

which is straightforward. The other half of the unit law follows similarly. Associativity of vertical composition is by a similar argument using (2.5). Associativity of horizontal composition is clear from the definition of $\star$, as is $1_f \star 1_g = 1_{f \circ g}$. □

Remark 2.3.5 The situation of Proposition 2.3.4 is to be contrasted with that of topological spaces and homotopies of continuous maps where it is necessary first to quotient by the existence of appropriate 3-cells (homotopies of homotopies). This points the way for further work on the notion of a *weak* interval.

When $\mathcal{C}$ is a cocartesian ccc with an interval object I we write $(\mathcal{C}, I)$ for the 2-category described in Proposition 2.3.4.

Using the 2-categorical structure of $(\mathcal{C}, I)$ we may now easily verify the "three-for-two" axiom.

Corollary 2.3.6 *If $\mathcal{C}$ is a cocartesian ccc with an interval object I, then the weak equivalences as defined above satisfy the "three-for-two" axiom.*

Proof Let maps $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be given. First, assume $g \circ f$ and g are weak equivalences. As such, there exist maps $g' : B \longrightarrow C$ and $h : C \longrightarrow A$ together with homotopies $\gamma_0 : g' \circ g \Longrightarrow 1_B$, $\gamma_1 : g \circ g' \Longrightarrow 1_C$, $\eta_0 : h \circ (g \circ f) \Longrightarrow 1_A$ and $\eta_1 : (g \circ f) \circ h \Longrightarrow 1_C$. Define $f' := h \circ g$ and observe that $\eta_0 : f' \circ f \Longrightarrow 1_A$. To construct the other required 2-cell $f \circ f' \Longrightarrow 1_B$ note that, since $(\mathcal{C}, I)$ is a 2-category, we need only provide a "pasting-diagram" (cf. [11]):

$$\begin{tikzcd}
 & & A \arrow[dr, "f"] \arrow[d, phantom, "\eta_1\Downarrow"] & & \\
 & C \arrow[ur, "h"] \arrow[dr, "1_C"'] & {} & B \arrow[dl, "g"'] \arrow[dr, "1_B"] \arrow[d, phantom, "\Downarrow\bar{\gamma}_0"] & \\
B \arrow[ur, "g"] \arrow[rrrr, bend right, "1_B"'] & & C \arrow[rr, "g'"'] \arrow[d, phantom, "\gamma_0\Downarrow"] & {} & B \\
 & & {} & &
\end{tikzcd}$$

where $\bar{\gamma_0} : 1_B \Longrightarrow g' \circ g$ is the "reverse homotopy" as discussed in Remark 2.2.4. The other two cases are similarly verified. □

2.4 Path objects

In order to use intervals to construct a path object as the interpretation of the identity type, we need to know that $A^I \longrightarrow A \times A$ is a fibration.

Definition 2.4.1 An object A of $\mathcal{C}$ is *contractible* if and only if the canonical map $!_A : A \longrightarrow 1$ is a weak equivalence. A subobject $m : S \rightarrowtail A$ is a *strong deformation retract of* A if there exists a retraction $r : A \longrightarrow S$ and a homotopy $\eta : m \circ r \Longrightarrow 1_A$ such that the following diagram commutes:

$$\begin{tikzcd}
S \times I \arrow[r, "m \times 1_I"] \arrow[d, "\pi_S"'] & A \times I \arrow[d, "\eta"] \\
S \arrow[r, "m"'] & A.
\end{tikzcd}$$

Proposition 2.4.2 (Warren) *The following are equivalent:*

1. *For any object A of $\mathcal{C}$, the map $\iota : A^I \longrightarrow A \times A$ defined by $\iota := \langle A^{\perp}, A^{\top} \rangle$ is a fibration.*

2. *The interval I is contractible in the strong sense that the map $\perp : 1 \longrightarrow I$ is a strong deformation retract of I.*

3. There exists a binary operation $\overline{\wedge} : I \times I \longrightarrow I$ such that the following diagrams commute:

$$\begin{tikzcd} I \arrow[r, "I_0"] \arrow[d, "I^0"'] \arrow[dr, "\perp\circ !"] & I \times I \arrow[d, "\overline{\wedge}"] \\ I \times I \arrow[r, "\overline{\wedge}"'] & I, \end{tikzcd}$$

and:

$$\begin{tikzcd} I \arrow[rr, "I_1"] \arrow[dr, "1_I"'] & & I \times I \arrow[dl, "\overline{\wedge}"] \\ & I, & \end{tikzcd}$$

where $I^0 := \langle \perp\circ !, 1_I \rangle$. I.e., internally:

$$\begin{aligned} x \mathbin{\overline{\wedge}} \perp &= \perp \\ &= \perp \mathbin{\overline{\wedge}} x, \end{aligned}$$

and:

$$x \mathbin{\overline{\wedge}} \top = x,$$

for $x : I$.

Proof (2) and (3) are clearly equivalent. To see that (1) implies (3) notice that since $\iota : I^I \longrightarrow I \times I$ is a fibration there exists a lift $\lambda : I \longrightarrow I^I$ as indicated in the following diagram:

$$\begin{tikzcd} 1 \arrow[r, "k_\perp"] \arrow[d, "\perp"'] & I^I \arrow[d, "\iota"] \\ I \arrow[r, "I^0"'] \arrow[ur, dotted, "\lambda"] & I \times I. \end{tikzcd}$$

The desired map $\overline{\wedge}$ is then defined to be the exponential transpose of λ.

Now to see that (3) implies (1), assume that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "Y_0"'] & A^I \arrow[d, "\iota"] \\ Y \times I \arrow[r, "h"'] & A \times A \end{tikzcd} \tag{2.6}$$

and let $\lambda : I \longrightarrow I^I$ be the transpose of $\overline{\wedge}$.

Write $\alpha : Y \times I \longrightarrow A$ for $\pi_1 \circ h$, $\beta : Y \times I \longrightarrow A$ for $\pi_2 \circ h$ and $\gamma : Y \times I \longrightarrow A$ for the transpose of $g : Y \longrightarrow A^I$. Finally, define maps $\hat{\alpha}, \hat{\gamma} : Y \times I \times I \rightrightarrows A$ as follows:

$$\begin{aligned} \hat{\alpha} &:= \alpha \circ (1_Y \times \overline{\wedge}) \circ (1_Y \times 1_I \times \rho), \text{ and} \\ \hat{\gamma} &:= \gamma \circ (\pi_1 \times \pi_3). \end{aligned}$$

Then we may form the vertical composition $(\hat{\gamma} \cdot \hat{\alpha})$ as indicated in the following diagram:

$$\begin{tikzcd}
Y \times I \times 1 \arrow[r] \arrow[d, "1_Y \times 1_I \times \top"'] & Y \times I \times I \arrow[d, "1_Y \times 1_I \times \uparrow"'] \arrow[ddr, bend left, "\hat{\gamma}"] & \\
Y \times I \times I \arrow[r, "1_Y \times 1_I \times \downarrow"'] \arrow[drr, bend right, "\hat{\alpha}"'] & Y \times I \times I_2 \arrow[dr, dotted, "{c[\hat{\gamma}, \hat{\alpha}]}"] & \\
 & & A
\end{tikzcd}$$

where the canonical map $c[\hat{\gamma}, \hat{\alpha}]$ exists since the square is a pushout and:

$$\begin{aligned} \hat{\gamma} \circ (1_Y \times 1_I \times \bot) &= \gamma \circ Y_0 \\ &= \alpha \circ Y_0 \\ &= \alpha \circ (1_Y \times \overline{\wedge}) \circ (1_Y \times 1_I \times \bot) \\ &= \alpha \circ (1_Y \times \overline{\wedge}) \circ (1_Y \times 1_I \times \rho) \circ (1_Y \times 1_I \times \top) \\ &= \hat{\alpha} \circ (1_Y \times 1_I \times \top). \end{aligned}$$

Here the second identity is by (2.6).

Now, where $\hat{\beta} : Y \times I \times I \longrightarrow A$ is defined as $\hat{\beta} := \beta \circ (1_Y \times \overline{\wedge})$ we may again form the vertical composite $(\hat{\beta} \cdot (\hat{\gamma} \cdot \hat{\alpha}))$ since:

$$\begin{aligned} \hat{\beta}(1_Y \times 1_I \times \bot) &= \beta \circ Y_0 \\ &= \gamma \circ Y_1 \\ &= \hat{\gamma} \circ (1_Y \times 1_I \times \top) \\ &= c[\hat{\gamma}, \hat{\alpha}] \circ (1_Y \times 1_I \times (\uparrow \circ \top)) \\ &= (\hat{\gamma} \cdot \hat{\alpha}) \circ (1_Y \times 1_I \times \top), \end{aligned}$$

where the second identity is again by (2.6) and the final identity is by (2.3).

Write $\delta : Y \times I \longrightarrow A^I$ for the transpose of $(\hat{\beta} \cdot (\hat{\gamma} \cdot \hat{\alpha}))$. We claim that δ is the required lift. First, that $\iota \circ \delta = h$ is straightforward using the definition of δ. Secondly, to see that $\delta \circ Y_0 = g$ notice that:

$$(\hat{\beta} \cdot (\hat{\gamma} \cdot \hat{\alpha})) \circ (Y_0 \times 1_I) = c[\hat{\beta} \circ (Y_0 \times 1_I), (\hat{\gamma} \cdot \hat{\alpha}) \circ (Y_0 \times 1_I)] \circ (1_Y \times *).$$

Also notice that:

$$\begin{aligned} \hat{\beta} \circ (Y_0 \times 1_I) &= \beta \circ (1_Y \times \overline{\wedge}) \circ (1_Y \times I^0) \\ &= \beta \circ (1_Y \times \perp \circ !_I). \end{aligned}$$

But $\beta \circ (1_Y \times \perp \circ !_I)$ is an identity 2-cell $1_{\beta \circ Y_0} : \beta \circ Y_0 \Longrightarrow \beta \circ Y_0$. A similar calculation shows that $\hat{\alpha} \circ (Y_0 \times 1_I)$ is the identity 2-cell $1_{\alpha \circ Y_0}$. Combining this with the foregoing we obtain:

$$\begin{aligned} (\hat{\beta} \cdot (\hat{\gamma} \cdot \hat{\alpha})) \circ (Y_0 \times 1_I) &= (\hat{\gamma} \cdot \hat{\alpha}) \circ (Y_0 \times 1_I) \\ &= c[\hat{\gamma} \circ (Y_0 \times 1_I), \hat{\alpha} \circ (Y_0 \times 1_I)] \\ &= \hat{\gamma} \circ (Y_0 \times 1_I) \\ &= \gamma. \end{aligned}$$

Therefore $g = \delta \circ Y_0$, as required. □

Observe that the proof of Proposition 2.4.2 uses the fact that the interval is *strict* in the sense that all of the cocategory equations commute "on the nose" and not up to the existence of higher dimensional isomorphisms. The intuition behind this proof is that $\overline{\wedge} : I \times I \longrightarrow I$ is just *rescaling*. I.e., we think of the action of $\overline{\wedge}$ as multiplication:

$$x \,\overline{\wedge}\, y \quad := \quad (xy),$$

for x, y real numbers in the closed unit interval. Of course, this intuition should not be taken too seriously since $\overline{\wedge}$ need not even be commutative.

Corollary 2.4.3 *If $\mathcal{C}$ satisfies the equivalent conditions from Proposition 2.4.2, then, for any object A, the "constant loop" (or "diagonal" or "reflexivity") map $r : A \longrightarrow A^I$ is a strong deformation retract of A^I.*

Proof Clearly r is a section of $A^{\perp} : A^I \longrightarrow A$. The required homotopy $\eta : r \circ A^{\perp} \Longrightarrow 1_{A^I}$ is constructed as the transpose of the composite:

$$A^I \times I \times I \xrightarrow{1_{A^I} \times \overline{\wedge}} A^I \times I \xrightarrow{\mathrm{ev}} A.$$

Then η is a homotopy $r \circ A^{\perp} \Longrightarrow 1_{A^I}$ by definition of $\overline{\wedge}$. Finally, η is a strong deformation retract since $\mathrm{ev} \circ (r \times 1_I) = \pi_A$. □

Definition 2.4.4 A cocartesian ccc $\mathcal{C}$ with an interval object I has *(interval) identity types induced by* I if it satisfies the following conditions:

(I0) I satisfies any of the equivalent conditions from Proposition 2.4.2. I.e., $\perp : 1 \longrightarrow I$ is a strong deformation retract of I.

(I1) For every object A, the constant loop map $r : A \longrightarrow A^I$, which arises as the transpose of the projection $A \times I \longrightarrow A$, has the left lifting property with respect to all fibrations:

$$r \perp \mathfrak{F}.$$

Example 2.4.5 The category **Gpds** of small groupoids clearly satisfies **(I1)** since it possesses a model structure for which the map $r : \mathbf{G} \longrightarrow \mathbf{G}^{\mathbf{I}}$ is an acyclic cofibration (i.e., a categorical equivalence which is injective on objects). Moreover, $\bar{\wedge} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ may be defined by setting $\perp \bar{\wedge} x = x \bar{\wedge} \perp = \perp$ for any object x of $\mathbf{I}$ and $\top \bar{\wedge} x = x \bar{\wedge} \top = x$ for any object x of $\mathbf{I}$. Therefore, **Gpds** possesses identity types induced by $\mathbf{I}$.

It should be clear from the proof of Proposition 1.4.1 that if $\mathcal{C}$ has interval identity types induced by I, then it also satisfies all of the type theoretic rules governing propositional identity types. This is convenient since it should be easier to prove that a category has interval identity types than that it is a model category. Of course, this axiomatization is not entirely satisfactory as it stands, since condition **(I1)** still needs to be analyzed further, and one feels that its connection with model categories should be made clearer. While we have not yet found a satisfactory (i.e., sufficiently non-trivial) answer to this question we nonetheless consider in Appendix A.2 below one notion of cofibration in a category satisfying the conditions of Definition 2.4.4.

2.5 Groupoids

One of the principal examples of a category which has identity types induced by an interval is the category **Gpds** of small groupoids and functors between them. The interval is, as mentioned above, the connected two element groupoid $\mathbf{I}$. One particularly nice feature of **Gpds** is that the model category structure coincides with that induced by $\mathbf{I}$. It is to proving this which we now turn. First, recall that in the Quillen homotopy structure for Groupoids the weak equivalences are just ordinary categorical equivalences (cf. [10]). I.e., those maps for which there exists a pseudo-inverse.

Lemma 2.5.1 *Given functors $f, g : \mathbf{G} \rightrightarrows \mathbf{H}$ in* **Gpds**, *there is an isomorphism:*

$$\mathrm{Homotopies}(f, g) \quad \cong \quad \mathrm{Nat}(f, g),$$

between the collection of homotopies from f to g and the collection of natural transformations from f to g.

Proof First, given a homotopy $\eta : f \Longrightarrow g$ and an object a of $\mathbf{G}$, define $\hat{\eta}_a : f(a) \longrightarrow g(a)$ to be the map:

$$\begin{tikzcd} \eta(a, \perp) \arrow[r, "\eta(a,u)"] \arrow[d, phantom, "\shortparallel"] & \eta(a, \top) \arrow[d, phantom, "\shortparallel"] \\ f(a) & g(a), \end{tikzcd}$$

where u is the map $\perp \longrightarrow \top$ as defined in Example 2.2.2. This assignment determines a natural transformation $\hat{\eta}$ from f to g.

Alternatively, given a natural transformation γ from f to g define a homotopy $\check{\gamma} : \mathbf{G} \times \mathbf{I} \longrightarrow \mathbf{H}$ by the following assignments:

$$\begin{aligned} \check{\gamma}(a, \perp) &:= f(a), \\ \check{\gamma}(a, \top) &:= g(a), \\ \check{\gamma}(a, u) &:= \gamma_a, \text{ and} \\ \check{\gamma}(a, d) &:= \gamma_a^{-1}, \end{aligned}$$

where a is an object of $\mathbf{G}$ and where the definition extended to maps in $\mathbf{G}$ in the obvious way.

Finally, $\hat{}$ and $\check{}$ are clearly inverse to one another by definition. $\square$

In order to make the connection between these two structures on the category of groupoids explicit it is natural to compare the corresponding 2-categories $(\mathbf{Gpds}, \mathbf{I})$ and **Gpds**, where the 2-cells of the latter are natural transformations.

Remark 2.5.2 It is useful to note that in **Gpds** if we are given two composable homotopies:

$$\begin{tikzcd} \mathbf{G} \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & \mathbf{H} \end{tikzcd} \tag{2.7}$$

and an object a of $\mathbf{G}$, then:

$$(\beta \cdot \alpha)(a, u) \quad = \quad \beta(a, u) \circ \alpha(a, u), \tag{2.8}$$

since:

$$\begin{aligned} c[\beta, \alpha] \circ (1_{\mathbf{G}} \times *)(a, u) &= c[\beta, \alpha](a, u_{\uparrow} \circ u_{\downarrow}) \\ &= c[\beta, \alpha](a, u_{\uparrow}) \circ c[\beta, \alpha](a, u_{\downarrow}). \end{aligned}$$

The corresponding remark also applies to $(\beta \cdot \alpha)(a, d)$.

Proposition 2.5.3 *There exists an isomorphism of 2-categories:*

$$(\mathbf{Gpds}, \mathbf{I}) \quad \cong \quad \mathbf{Gpds}.$$

PROOF First, we define 2-functors (cf. Definition A.3.2):

$$\begin{aligned} (\mathbf{Gpds}, \mathbf{I}) &\xrightarrow{F} \mathbf{Gpds}, \text{ and} \\ \mathbf{Gpds} &\xrightarrow{G} (\mathbf{Gpds}, \mathbf{I}) \end{aligned}$$

as follows. Both F and G are defined to be the identity on objects and arrows. On 2-cells:

$$\begin{aligned} F(\eta) &:= \hat{\eta}, \text{ and} \\ G(\gamma) &:= \check{\gamma}, \end{aligned}$$

where $\hat{\cdot}$ and $\check{\cdot}$ are as defined in Lemma 2.5.1. Now we must show that F and G preserve vertical and horizontal composition of 2-cells. Let the data of Diagram (2.7) from Remark 2.5.2 above be given where the 2-cells are in $(\mathbf{Gpds}, \mathbf{I})$ and let an object a of $\mathbf{G}$ be given. We must show that:

$$(\widehat{\beta \cdot \alpha})_a \quad = \quad (\hat{\beta})_a \circ (\hat{\alpha})_a.$$

To this end observe that:

$$\begin{aligned} (\widehat{\beta \cdot \alpha})_a &= \beta \cdot \alpha(a, u) \\ &= \beta(a, u) \circ \alpha(a, u) \\ &= (\hat{\beta})_a \circ (\hat{\alpha})_a, \end{aligned}$$

as required. Next we require that if the 2-cells of Diagram (2.7) are 2-cells of $\mathbf{Gpds}$, then:

$$G(\beta \cdot \alpha) \quad = \quad \check{\beta} \cdot \check{\alpha}. \tag{2.9}$$

A similar calculation to that just given, now using the fact that the right-hand side of (2.9) is equal to $c[\check{\beta}, \check{\alpha}] \circ (1_{\mathbf{G}} \times *)$, shows that this equation also holds.

For horizontal composition, assume given the following data:

$$\begin{tikzcd} \mathbf{G} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathbf{H} \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathbf{K}, \end{tikzcd} \tag{2.10}$$

with the 2-cells from $(\mathbf{Gpds}, \mathbf{I})$. To show that $F(\beta \star \alpha) = F(\beta) \star F(\alpha)$, assume given an object a of $\mathbf{G}$ and note that:

$$\begin{aligned} (\hat{\beta} \star \hat{\alpha})_a &= \hat{\beta}_{g(a)} \circ h(\hat{\alpha}_a) \\ &= \beta(g(a), u) \circ h(\alpha(a, u)) \\ &= \beta(\alpha(a, \top), u) \circ \beta(\alpha(a, u), \bot) \\ &= \beta(\alpha(a, u), u) \\ &= (\widehat{\beta \star \alpha})_a, \end{aligned}$$

as required.

Now suppose the data of (2.10) is taken in $\mathbf{Gpds}$ to show that $G(\beta \star \alpha) = G(\beta) \star G(\alpha)$. To see that this is the case notice that if we are given an object a of $\mathbf{G}$, then:

$$\begin{aligned} G(\beta) \star G(\alpha)(a, u) &= G(\beta)\big(G(\alpha)(a, u), u\big) \\ &= G(\beta)(\alpha_a, u) \\ &= \beta_{g(a)} \circ h(\alpha_a) \\ &= (\beta \star \alpha)_a \\ &= G(\beta \star \alpha)(a, u), \end{aligned}$$

as required. The other case is similar.

Finally, identity maps are preserved by definition and, by Lemma 2.5.1, F and G are inverses. $\square$

Corollary 2.5.4 *A functor $f : \mathbf{G} \longrightarrow \mathbf{H}$ is a weak equivalence in the sense of Definition 2.2.3 if and only if it is a categorical equivalence.*

Proof First, assume f is a weak equivalence as defined in Definition 2.2.3. As such, there exists a map $f' : \mathbf{H} \longrightarrow \mathbf{G}$ together with homotopies η :

$f \circ f' \Longrightarrow 1_{\mathbf{H}}$ and $\gamma : f' \circ f \Longrightarrow 1_{\mathbf{G}}$. But then, where F and G are the 2-functors defined in Proposition 2.5.3, $F(\eta)$ is a natural transformation from $f \circ f'$ to $1_{\mathbf{H}}$ and similarly for $F(\gamma)$. Moreover, both of these are natural isomorphisms since all maps in **Gpds** are isomorphisms.

Similarly, given a categorical equivalence $f : \mathbf{G} \longrightarrow \mathbf{H}$ together with its pseudo-inverse $f' : \mathbf{H} \longrightarrow \mathbf{G}$ and the witnessing natural isomorphisms $\phi : f \circ f' \cong 1_{\mathbf{H}}$ and $\psi : f' \circ f \cong 1_{\mathbf{G}}$ we obtain the required homotopies by simply applying the 2-functor G. □

Definition 2.5.5 A map $p : \mathbf{E} \longrightarrow \mathbf{B}$ of groupoids is a *Grothendieck fibration* if for any object e of $\mathbf{E}$ and arrow $j : b \longrightarrow p(e)$ in $\mathbf{B}$, there exists a map $\tilde{j} : \tilde{b} \longrightarrow e$ in $\mathbf{E}$ such that $p(\tilde{j}) = j$.

Lemma 2.5.6 *A map $p : \mathbf{E} \longrightarrow \mathbf{B}$ of groupoids is a Grothendieck fibration if and only if it is a fibration in the sense of Definition 2.2.3.*

Proof Let a Grothendieck fibration $p : \mathbf{E} \longrightarrow \mathbf{B}$ together with a commutative diagram:

$$\begin{tikzcd} \mathbf{G} \arrow[r, "h"] \arrow[d, "\mathbf{G}_0"'] & \mathbf{E} \arrow[d, "p"] \\ \mathbf{G} \times \mathbf{I} \arrow[r, "k"'] & \mathbf{B} \end{tikzcd}$$

be given. Fix an object a of $\mathbf{G}$. Then $p(h(a)) = k(a, \perp)$ and $k(a, d) : k(a, \top) \longrightarrow k(a, \perp)$. Thus, since p is a Grothendieck fibration, there exists a lift $j : b \longrightarrow h(a)$ in $\mathbf{E}$ such that $p(j) = k(a, d)$. As such, we define a functor $g : \mathbf{G} \times \mathbf{I} \longrightarrow \mathbf{E}$ by setting $g(a, \perp) := h(a)$, $g(a, \top) := b$, $g(a, u) := j^{-1}$ and $g(a, d) := j$. Therefore, g is the required lift.

Conversely, suppose p is a fibration induced by the interval $\mathbf{I}$ and let an object e of $\mathbf{E}$ together with a map $j : b \longrightarrow p(e)$ in $\mathbf{B}$ be given. Then the following diagram commutes:

$$\begin{tikzcd} \mathbf{1} \arrow[r, "e"] \arrow[d, "\perp"'] & \mathbf{E} \arrow[d, "p"] \\ \mathbf{I} \arrow[r, "h"'] & \mathbf{B}, \end{tikzcd}$$

where $h : \mathbf{I} \longrightarrow \mathbf{B}$ is defined by setting $h(\perp) := p(e)$, $h(\top) := b$, $h(d) := j$ and $h(u) := j^{-1}$. Therefore, since p is a fibration there exists a lift $g : \mathbf{I} \longrightarrow \mathbf{E}$. Therefore, $g(d)$ is the required lift of $j : b \longrightarrow p(e)$ in $\mathbf{E}$. □

The following lemma establishes a special property of cofibrations in **Gpds**. For the definition of *Hurewicz cofibrations* and $\mathfrak{C}_H$ the reader should refer to Definition A.2.1. Recall that cofibrations in **Gpds** are functors injective on objects.

Lemma 2.5.7 *In* **Gpds**, $\mathfrak{C}_H = \mathfrak{C}$. *I.e., Hurewicz and ordinary cofibrations coincide.*

PROOF By (3) of Lemma A.2.2 we have that $\mathfrak{C} \subseteq \mathfrak{C}_H$. Therefore, suppose a map of groupoids $i : \mathbf{G} \longrightarrow \mathbf{H}$ is a Hurewicz cofibration and let objects a and b of $\mathbf{G}$ be given such that $i(a) = i(b)$.

Consider the following diagram:

$$\begin{tikzcd} \mathbf{G} \arrow[r, "f"] \arrow[d, "i"'] & \mathbf{I}^{\mathbf{I}} \arrow[d, "\mathbf{I}^{\perp}"] \\ \mathbf{H} \arrow[r, "g"'] & \mathbf{I} \end{tikzcd} \tag{2.11}$$

where:

$$f(x) \quad := \quad \begin{cases} 1_\top & \text{if } x = a; \\ d & \text{if } x \cong a \text{ but } x \neq a; \text{ and} \\ 1_\perp & \text{otherwise;} \end{cases}$$

where x is an object of $\mathbf{G}$. On arrows:

$$f(\alpha : x \longrightarrow y) \quad := \quad \begin{cases} 1_{(1_\perp)} & \text{if } x, y \ncong a; \\ 1_d & \text{if } x, y \cong a \text{ and } x \neq a \neq y; \\ 1_{(1_\top)} & \text{if } x = y = a; \\ (1_\top, u) & \text{if } x \cong a,\ x \neq a \text{ and } y = a; \text{ and} \\ (1_\top, d) & \text{if } x = a,\ y \cong a \text{ but } y \neq a; \end{cases}$$

where $\alpha : x \longrightarrow y$ is an arrow of $\mathbf{G}$. Let C_a be the set of objects of $\mathbf{G}$ isomorphic to a (i.e., the "connected component" of a). Let g be defined by:

$$g(x) \quad := \quad \begin{cases} \top & \text{if } x \in i(C_a); \\ \perp & \text{otherwise;} \end{cases}$$

for x an object of $\mathbf{H}$ and:

$$g(\alpha : x \longrightarrow y) \quad := \quad \begin{cases} 1_\perp & \text{if neither } x \text{ nor } y \text{ is in } i(C_a); \\ u & \text{if } x \notin i(C_a), \text{ but } y \in i(C_a); \\ d & \text{if } x \in i(C_a), \text{ but } y \notin i(C_a); \text{ and} \\ 1_\top & \text{if } x, y \in i(C_a); \end{cases}$$

where $\alpha : x \longrightarrow y$ is an arrow in $\mathbf{H}$.

Given these definitions f and g are functorial and (2.11) commutes. Therefore, since i is a Hurewicz cofibration there exists a lift $\hat{f}$ making the following diagram commute:

$$\begin{tikzcd} \mathbf{G} \arrow[r, "f"] \arrow[d, "i"'] & \mathbf{I}^{\mathbf{I}} \arrow[d, "\mathbf{I}^{\perp}"] \\ \mathbf{H} \arrow[r, "g"'] \arrow[ur, dotted, "\hat{f}"] & \mathbf{I}. \end{tikzcd}$$

Thus:

$$\begin{aligned} f(a) &= \hat{f} \circ i(a) \\ &= \hat{f} \circ i(b) \\ &= f(b); \end{aligned}$$

but, by definition of f, then $a = b$. $\square$

Appendix A

Assorted Additional Material

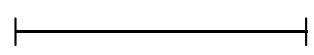
Below we collect both a schematic picture of the definition of interval object and, for the convenience of the reader, the definition of 2-category.

A.1 A schematic picture of the definition of a interval object

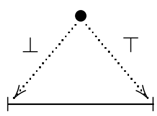
We will now give a brief presentation of the "intended picture" of cocategory objects which should help the reader understand the intuition a little better (this picture is in some sense just a way of illustrating the cocategory object in **Gpds** discussed below). To begin with, we will regard C_0 as a single point:

$$\bullet,$$

and C_1 will be regarded as the "unit interval":

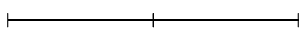


The maps $\perp$ and $\top$ then are simply points of the interval:

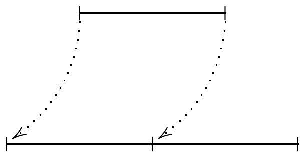


where $\perp$ is identified with the "bottom" end of the interval and $\top$ is identified with the "top" end.

C_2 is then, by definition, the result of gluing the interval to itself by identifying the top and bottom:

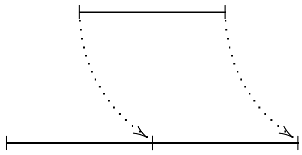


and the maps $\downarrow, \uparrow \colon C_1 \rightrightarrows C_2$ have the actions illustrated as follows:



$$\begin{tikzcd} C_1 \arrow[d, "\downarrow"] \\ C_2 \end{tikzcd}$$

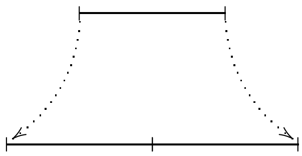
and:



$$\begin{tikzcd} C_1 \arrow[d, "\uparrow"] \\ C_2 \end{tikzcd}$$

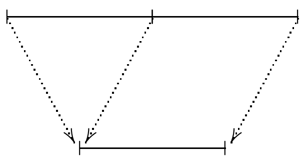
The point $\downarrow \circ \top = \uparrow \circ \bot$ may (in some sense) be identified with the midpoint of C_2.

The cocomposition $* : C_1 \longrightarrow C_2$ is then the "magnification" operation:



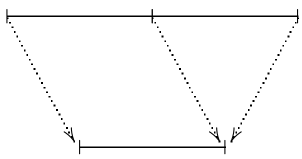
$$\begin{tikzcd} C_1 \arrow[d, "*"] \\ C_2 \end{tikzcd}$$

The maps $i_0, i_1 : C_2 \rightrightarrows C_1$ mentioned in the fourth axiom for cocategory objects have the action, in this case, of collapsing the initial segment to $\bot$ and collapsing the final segment to $\top$, respectively. This is illustrated as follows:



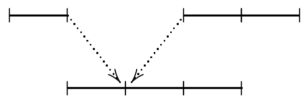
$$\begin{tikzcd} C_2 \arrow[d, "i_0"] \\ C_1 \end{tikzcd}$$

and

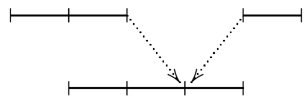


$$\begin{tikzcd} C_2 \arrow[d, "i_1"] \\ C_1 \end{tikzcd}$$

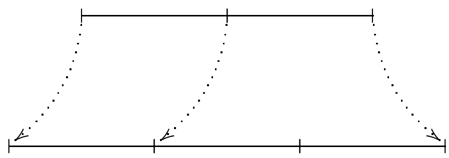
The construction of C_3 may be visualized as:



or

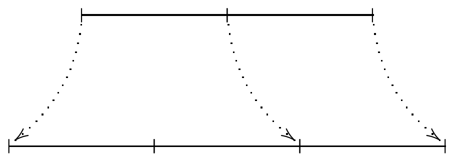


The maps $t_0, t_1 : C_2 \rightrightarrows C_3$ are then given schematically by:



$$\begin{tikzcd} C_2 \arrow[d, "t_0"] \\ C_3 \end{tikzcd}$$

and:



$$\begin{tikzcd} C_2 \arrow[d, "t_1"] \\ C_3 \end{tikzcd}$$

A.2 Cofibrations

Throughout this section we assume that $\mathcal{C}$ is a cocartesian ccc with interval object I satisfying condition **(I0)** of Definition 2.4.4 unless otherwise stated. We have already defined the fibrations $\mathfrak{F}$ in $\mathcal{C}$ to be the Hurewicz fibrations and the weak equivalences $\mathfrak{W}$ to be the homotopy equivalences. In this section we study two kinds of cofibrations in such a category and present two small results about cofibrations. The proofs of these results are straightforward and are, in the interest of space, therefore omitted.

The "only" real candidate for the cofibrations is then the collection of maps ${}^{\perp}(\mathfrak{F} \cap \mathfrak{W})$ which have the left lifting property with respect to acyclic fibrations. For practical purposes it will be convenient to have a better description of the cofibrations (as, for instance, one does in the case of **Gpds**). For now we call a map a cofibration if and only if it is a member of this collection:

$$\mathfrak{C} \ := \ {}^{\perp}(\mathfrak{F} \cap \mathfrak{W}).$$

Definition A.2.1 A map $p : E \longrightarrow B$ is a *trivial fibration* if and only if it has the right lifting property with respect to cofibrations. As such, $\mathfrak{C}^{\perp}$ is the collection of trivial fibrations.

A map $i : A \longrightarrow X$ is a *Hurewicz cofibration* if and only if it has the *homotopy extension property* (HEP). I.e., for any commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B^I \arrow[d, "B^\perp"] \\ X \arrow[r] & B \end{tikzcd}$$

there exists a diagonal filler. We write $\mathfrak{C}_H$ for the collection of Hurewicz cofibrations.

Lemma A.2.2 *Given the foregoing definition of* $\mathfrak{C}$ *and* $\mathfrak{C}_H$ *in* $\mathcal{C}$ *the following hold.*

1. *Both* $\mathfrak{C}$ *and* $\mathfrak{C}_H$ *are stable under composition. Moreover, both collections contain all isomorphisms.*
2. *Every canonical map* $0 \longrightarrow A$ *is a Hurewicz cofibration. I.e., all objects are Hurewicz cofibrant.*
3. $\mathfrak{C} \subseteq \mathfrak{C}_H$.
4. *(Cobase change) Both* $\mathfrak{C}$ *and* $\mathfrak{C}_H$ *are stable under cobase change.*
5. *All coproduct injections* $i_A : A \longrightarrow A + B$ *are in* $\mathfrak{C}_H$.

Lemma A.2.3 *All four classes of maps* $\mathfrak{F}$, $\mathfrak{W}$, $\mathfrak{C}$ *and* $\mathfrak{C}_H$ *satisfy the "retracts" axiom.*

A.3 2-Categories and their structure

A *2-category* $\mathfrak{C}$ consists of a category together with a notion of 'transformation' between the arrows of the category. As such, $\mathfrak{C}$ is comprised of three types of things. First, $\mathfrak{C}$ comes equipped, as does any category, with a collection $\mathfrak{C}_0$ of *objects* and a collection $\mathfrak{C}_1$ of *arrows*. These objects and arrows are endowed with the usual composition $f \circ g$, identities 1_C, domain $\mathrm{dom}(f)$ and codomain $\mathrm{cod}(f)$ operations satisfying the associative and unit laws. We write $f : A \longrightarrow B$ as usual to indicate that f is an arrow such that $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$. For a little variation we will sometimes call the objects *0-cells* and the arrows *1-cells*.

An abstract notion of "natural transformation" or "homotopy" between arrows in $\mathfrak{C}$ is given by a *third* collection $\mathfrak{C}_2$ of *2-cells*: α, β, et cetera.

Since 2-cells are to be thought of as natural transformations they also come equipped with domain and codomain operations:

$$\mathrm{dom}(-), \mathrm{cod}(-) : \mathfrak{C}_2 \rightrightarrows \mathfrak{C}_1.$$

Some authors call these "source" and "target" to distinguish them from the corresponding operations on 1-cells; however, we trust no confusion should result from this economy of notation. We write $\eta : f \Longrightarrow g$ to indicate that η is a 2-cell with $\mathrm{dom}(\eta) = f$ and $\mathrm{cod}(\eta) = g$. We will also sometimes indicate this diagrammatically by:

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\eta\Downarrow"] & D. \end{tikzcd}$$

Such diagrams have the advantage of providing information about the domain and codomain of f and g as well (in this example both maps have domain C and codomain D — and there are 2-cells only between 1-cells with the same domain and codomain.)

We also stipulate that $\mathfrak{C}$ come equipped with *two* forms of composition for 2-cells. These are called *vertical* and *horizontal* composition. Given 2-cells η and γ such that $\mathrm{cod}(\eta) = \mathrm{dom}(\gamma)$ we may form the vertical composite $\gamma \cdot \eta$. This vertical composite $\gamma \cdot \eta$ then has the same domain as η and the same codomain as γ. For instance, if $\eta : f \Longrightarrow g$ and $\gamma : g \Longrightarrow h$, then $\gamma \cdot \eta : f \Longrightarrow h$. This is illustrated diagrammatically as follows:

$$\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\eta\Downarrow"] \arrow[r, phantom, shift right=3, "\gamma\Downarrow"] & D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\gamma\cdot\eta \Downarrow"] & D. \end{tikzcd}$$

Moreover, for any arrow $f : C \longrightarrow D$ there exists an *identity 2-cell* $1_f : f \Longrightarrow f$ which is a unit for vertical composition.

For the second form of composition of 2-cells (horizontal composition), consider the following situation:

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\eta\Downarrow"] & D \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\gamma\Downarrow"] & E. \end{tikzcd}$$

The horizontal composite, written $\gamma * \eta$, is defined to be a 2-cell $h \circ f \Longrightarrow k \circ g$. That is, the horizontal composite $\gamma * \eta$ is defined when $\mathrm{cod}(\mathrm{dom}(\eta)) = \mathrm{dom}(\mathrm{dom}(\gamma))$. It is also required by the definition of 2-category that $1_g * 1_f = 1_{g \circ f}$ where $\mathrm{cod}(f) = \mathrm{dom}(g)$. Finally, we require that given:

$$\begin{tikzcd} C \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & D \arrow[r, bend left=60, "k"] \arrow[r, "l" description] \arrow[r, bend right=60, "m"'] \arrow[r, phantom, shift left=3, "\gamma\Downarrow"] \arrow[r, phantom, shift right=3, "\delta\Downarrow"] & E, \end{tikzcd}$$

the following *Interchange Law* holds:

$$(\delta * \beta) \cdot (\gamma * \alpha) \quad = \quad (\delta \cdot \gamma) * (\beta \cdot \alpha).$$

Putting the foregoing (informal) sketch together we obtain:

Definition A.3.1 A *2-category* $\mathfrak{C}$ consists of the following data:

Objects: A collection $\mathfrak{C}_0$ of *objects* (or *0-cells*) A, B et cetera.

Arrows: A collection $\mathfrak{C}_1$ of *arrows* (or *1-cells*) f, g et cetera.

2-cells: A collection $\mathfrak{C}_2$ of *2-cells* γ, δ, et cetera.

Domain and codomain: Maps $\mathrm{dom}_i(-), \mathrm{cod}_i(-) : \mathfrak{C}_{i+1} \rightrightarrows \mathfrak{C}_i$ for $i = 0, 1$ (we will omit these indices when no confusion will result).

Composition: Composition operations $(- \circ -)$ on 1-cells, and $(- \cdot -)$ and $(- * -)$ on 2-cells which are defined only when domains and codomains match in the sense that:

$$\begin{aligned} (f \circ g) \text{ is defined} &\quad \text{iff} \quad \mathrm{dom}(f) = \mathrm{cod}(g), \\ (\gamma \cdot \eta) \text{ is defined} &\quad \text{iff} \quad \mathrm{dom}(\gamma) = \mathrm{cod}(\eta), \text{ and} \\ (\gamma * \eta) \text{ is defined} &\quad \text{iff} \quad \mathrm{dom}(\mathrm{dom}(\gamma)) = \mathrm{cod}(\mathrm{cod}(\eta)). \end{aligned}$$

All three forms of composition are required to be associative.

Identities: Distinguished arrows $1_A : A \longrightarrow A$ for each object A in $\mathfrak{C}_0$ and 2-cells $1_f : f \Longrightarrow f$ for each arrow f in $\mathfrak{C}_1$. Therefore there are also 2-cells $1_{1_A} : 1_A \Longrightarrow 1_A$ for each object A in $\mathfrak{C}_0$. The arrows 1_A are required to be units for ordinary composition $\circ$, the 2-cells 1_f are required to be units for vertical composition $\cdot$, and the 1_{1_B} are required to be units for horizontal composition $*$. Moreover, we require that:

$$(1_g * 1_f) \quad = \quad 1_{g \circ f}.$$

Interchange law: Finally, we stipulate that the interchange law holds:

$$(\delta * \beta) \cdot (\gamma * \alpha) \quad = \quad (\delta \cdot \gamma) * (\beta \cdot \alpha),$$

whenever:

$$\begin{tikzcd} C \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & D \arrow[r, bend left=50, "k"] \arrow[r, "l" description] \arrow[r, bend right=50, "m"'] \arrow[r, phantom, shift left=3, "\gamma\Downarrow"] \arrow[r, phantom, shift right=3, "\delta\Downarrow"] & E, \end{tikzcd}$$

Once one has defined 2-category it is natural to define a notion of *2-functor* thereby permitting us to compare 2-categories.

Definition A.3.2 Given 2-categories $\mathfrak{C}$ and $\mathfrak{D}$ a *(strict) 2-functor* $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ consists of maps $F_0 : \mathfrak{C}_0 \longrightarrow \mathfrak{D}_0$, $F_1 : \mathfrak{C}_1 \longrightarrow \mathfrak{D}_1$ and $F_2 : \mathfrak{C}_2 \longrightarrow \mathfrak{D}_2$, which preserve (both forms of) identities and (all three forms of) composition.

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