Coalgebras in a category of classes

M. A. Warren

January 7, 2006

Abstract

In this paper the familiar construction of the category of coalgebras for a cartesian comonad is extended to the setting of "algebraic set theory". In particular, it is shown that, under suitable assumptions, several kinds of categories of classes are stable under the formation of coalgebras for a cartesian comonad, internal presheaves and comma categories.

In this note we extend the familiar fact that if $\mathcal{E}$ is a topos and G is a cartesian comonad on $\mathcal{E}$, then the category of coalgebras $\mathcal{E}_G$ is also a topos to the setting of "algebraic set theory" (cf. [6]). Specifically, in Section 2 we prove that if $\mathcal{C}$ is what we call a basic category of classes and G is a cartesian comonad which preserves small maps, then the category of coalgebras $\mathcal{C}_G$ is also a basic category of classes. In Sections 3 and 4 these results are extended to categories of classes with additional special properties. In Section 5 we apply these results to the study of internal presheaves and prove that all three types of categories of classes are stable under the construction of internal presheaves. In Section 6 we verify that they are also stable under the construction of comma categories. Appendix A contains definitions of the kinds of categories of classes being considered and useful, although not entirely novel, facts about the category of coalgebras are compiled in Appendices B through D. Finally, the reader who is unfamiliar with the kinds of categories being studied may consult [1], [2] or [8].

Acknowledgements. I thank Steve Awodey, Henrik Forssell, Nicola Gambino, Ivar Rummelhoff and Alex Simpson for useful discussion of these and closely related results.

1 Small maps in $\mathcal{C}_G$

Recall that a *comonad* on a category $\mathcal{C}$ consists of an endofunctor $G : \mathcal{C} \longrightarrow \mathcal{C}$ together with two natural transformations $\varepsilon : G \Longrightarrow 1_{\mathcal{C}}$ (*counit*) and $\delta : G \Longrightarrow G^2$ (*comultiplication*) such that:

$$\delta_{GC} \circ \delta_C = G\delta_C \circ \delta_C, \quad \text{and} \tag{1}$$

$$\varepsilon_{GC} \circ \delta_C = 1_{GC} = G\varepsilon_C \circ \delta_C \tag{2}$$

for any object C of $\mathcal{C}$. A *coalgebra* for a comonad (G, ε, δ) is then an object A of $\mathcal{C}$ together with a map $a : A \longrightarrow GA$ such that:

$$\begin{aligned} \varepsilon_A \circ a &= 1_A, \quad \text{and} && (3) \\ Ga \circ a &= \delta_A \circ a. && (4) \end{aligned}$$

Given coalgebras (A, a) and (B, b) a map $f : A \longrightarrow B$ in $\mathcal{C}$ is a *coalgebra homomorphism* when $b \circ f = Gf \circ a$. We denote the category consisting of coalgebras and coalgebra homomorphisms by $\mathcal{C}_G$. Explicit mention of the forgetful functor $U : \mathcal{C}_G \longrightarrow \mathcal{C}$ will often be omitted.

We begin by observing the following basic fact, the proof of which may be found in Appendix B:

Proposition 1.1 *If $\mathcal{C}$ is a positive heyting category and G is a cartesian comonad, then $\mathcal{C}_G$ is also a positive heyting category. I.e., axioms* **(C)** *of [1] are satisfied.*

Next, we define small maps in $\mathcal{C}_G$ and verify that they are a system of small maps (cf. Definition A.1).

Definition 1.2 A map $f : (A, a) \longrightarrow (B, b)$ in $\mathcal{C}_G$ is *small* if and only if $Uf : A \longrightarrow B$ is small in $\mathcal{C}$.

Proposition 1.3 *If $\mathcal{C}$ is a positive heyting category with a system of small maps and G is a cartesian comonad which preserves small maps, then $\mathcal{C}_G$ also satisfies* **(S1)**-**(S5)**.

Proof **(S1)** is by trivial. **(S2)** and **(S3)** are by the description of finite limits in $\mathcal{C}_G$. **(S4)** is by the fact that the forgetful functor $\mathcal{C}_G \longrightarrow \mathcal{C}$ preserves regular epimorphisms. Finally, **(S5)** is by the description of finite coproducts in $\mathcal{C}_G$. The reader is directed to Appendix B for more of the basic properties of $\mathcal{C}_G$ which are employed here. □

Observe that the assumption that G preserves small maps may be equivalently replaced with the assumption that it preserves small relations since G is cartesian.

2 Basic class structure and universal objects

Suppose now that $\mathcal{C}$ is has basic class structure and G is a cartesian comonad which preserves small maps. We first determine the powerobjects on $\mathcal{C}_G$. Let an object (A, a) of $\mathcal{C}_G$ be given. Because G preserves small relations there exists a unique classifying map $\tau : G\mathcal{P}_s\, A \longrightarrow \mathcal{P}_s\, GA$ such that the following diagram is a pullback:

$$\begin{array}{ccc} G\,\epsilon_A & \xrightarrow{\quad \tau' \quad} & \epsilon_{GA} \\ {\scriptstyle G\in_1} \big\downarrow & & \big\downarrow {\scriptstyle \in_2} \\ GA \times G\mathcal{P}_s\, A & \xrightarrow[1\times\tau]{} & GA \times \mathcal{P}_s\, GA \end{array}$$

Also, because (A, a) is a coalgebra the map a is a regular monomorphism and, *a fortiori*, a small map. As such, the powerobject functor has a contravariant action $a^* : \mathcal{P}_s\, GA \longrightarrow \mathcal{P}_s\, A$ along a.

In order to define the small powerobject of (A, a) first form the equalizer $i : E \rightarrowtail G\mathcal{P}_s\, A$ of $1_{G\mathcal{P}_s\, A}$ and $Ga^* \circ G\tau \circ \delta_{\mathcal{P}_s\, A}$ as illustrated in the following diagram.

$$\begin{array}{ccccccc} E & \xrightarrow{i} & G\mathcal{P}_s\, A & & \xrightarrow{1_{G\mathcal{P}_s\, A}} & & G\mathcal{P}_s\, A \\ & & & \searrow^{\delta_{\mathcal{P}_s\, A}} & & \nearrow_{Ga^*} & \\ & & & G^2\mathcal{P}_s\, A & \xrightarrow{G\tau} & G\mathcal{P}_s\, GA & \end{array}$$

Because G is cartesian, this definition induces a canonical map $e : E \longrightarrow GE$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \overset{i}{\rightarrowtail} & G\mathcal{P}_s\, A \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \delta_{\mathcal{P}_s\, A}} \\ GE & \underset{Gi}{\rightarrowtail} & G^2\mathcal{P}_s\, A. \end{array} \qquad (5)$$

since $\delta_{\mathcal{P}_s\, A} \circ i$ equalizes $G^2a^* \circ G^2\tau \circ G\delta_{\mathcal{P}_s\, A}$ and $1_{G^2\mathcal{P}_s\, A}$. Therefore, by Lemma B.5, (E, e) is a coalgebra and $i : (E, e) \rightarrowtail (GA, \delta_A)$ is a homomorphism.

Definition 2.1 Given (A, a) in $\mathcal{C}_G$, the *powerobject* $\mathcal{P}_s\,(A, a)$ of (A, a) in $\mathcal{C}_G$ is the object (E, e) as described above in (5).

Now, to find the membership relation on $(A, a)\times(E, e)$, define M as the following pullback:

$$\begin{array}{ccc} M & \overset{\mu'}{\rightarrowtail} & \epsilon_A \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \in_1} \\ A \times E & \underset{1_A\times(a^*\circ\tau\circ i)}{\rightarrowtail} & A \times \mathcal{P}_s\, A. \end{array}$$

Using the internal language of $\mathcal{C}$ we see that:

$$M \quad = \quad [\![x : A, y : E | x\ \epsilon_A\ a^* \circ \tau \circ i(y)]\!],$$

in $\mathrm{Sub}_{\mathcal{C}}(A \times E)$. We will henceforth adopt the following abbreviation:

$$\chi \quad := \quad a^* \circ \tau \circ i,$$

so that $\chi : E \longrightarrow \mathcal{P}_s\, A$.

Lemma 2.2 *There exists a canonical map* $m : M \longrightarrow GM$ *such that* (M, m) *is a coalgebra and* $\mu : (M, m) \rightarrowtail (A, a) \times (E, e)$ *is a homomorphism.*

Proof By Lemma B.5 it suffices to show that there exists a map $m : M \longrightarrow GM$ such that:

$$\begin{array}{ccc} M & \xrightarrow{\quad m \quad} & GM \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle G\mu} \\ A \times E & \xrightarrow[a \times e]{} & GA \times GE, \end{array} \tag{6}$$

commutes. Observe that the following square is a pullback since G is cartesian:

$$\begin{array}{ccc} GM & \xrightarrow{\quad \tau' \circ G\mu' \quad} & \epsilon_{GA} \\ {\scriptstyle G\mu}\big\downarrow & & \big\downarrow{\scriptstyle \in_2} \\ GA \times GE & \xrightarrow[1_{GA} \times (\tau \circ G\chi)]{} & GA \times \mathcal{P}_s\, GA. \end{array}$$

As such, it suffices to show that:

$$M \Vdash \alpha \;\epsilon_{GA}\; \tau \circ G\chi \circ \alpha', \tag{7}$$

where we have adopted the following abbreviations:

$$\begin{array}{rcl} \alpha & := & \pi_{GA} \circ (a \times e) \circ \mu = a \circ \pi_A \circ \mu, \quad \text{and} \\ \alpha' & := & \pi_{GE} \circ (a \times e) \circ \mu = e \circ \pi_E \circ \mu, \end{array}$$

so that $\alpha : M \longrightarrow GA$ and $\alpha' : M \longrightarrow GE$. To see that (7) holds consider the following equivalences:

$$\begin{array}{rll} M \Vdash \alpha \;\epsilon_{GA}\; \tau \circ G\chi \circ \alpha' & \text{iff} & M \Vdash \alpha \;\epsilon_{GA}\; \tau \circ Ga^* \circ G\tau \circ \delta_{\mathcal{P}_s A} \circ i \circ \pi_E \circ \mu \\ & \text{iff} & M \Vdash \alpha \;\epsilon_{GA}\; \tau \circ i \circ \pi_E \circ \mu \\ & \text{iff} & M \Vdash \pi_A \circ \mu \;\epsilon_A\; a^* \circ \tau \circ i \circ \pi_E \circ \mu \\ & \text{iff} & M \Vdash \pi_A \circ \mu \;\epsilon_A\; \chi \circ \pi_E \circ \mu, \end{array}$$

where the second equivalence is by the definition of i and the penultimate equivalence is by the definition of a^*. But $M \Vdash \pi_A \circ \mu \;\epsilon_A\; \chi \circ \pi_E \circ \mu$ holds trivially by definition of μ. Therefore, since $M \Vdash \alpha \;\epsilon_{GA}\; \tau \circ G\chi \circ \alpha'$, there exists a map $m : M \longrightarrow GM$ such that (6) commutes. □

Definition 2.3 Given (A, a) in $\mathcal{C}_G$, the *membership relation* $\epsilon_{(A,a)}$ for (A, a) is defined to be (M, m) as constructed in Lemma 2.2 above.

Notice that, by definition, $\mu : (M, m) \rightarrowtail (A, a) \times (E, e)$ is a small relation.

If $\rho : (R, r) \rightarrowtail (A, a) \times (B, b)$ is a small relation in $\mathcal{C}_G$, then ρ and $G\rho$ are both small relations in $\mathcal{C}$. As such, there exist unique classifying maps $\zeta : B \longrightarrow \mathcal{P}_s\, A$ and $\zeta' : GB \longrightarrow \mathcal{P}_s\, GA$ in $\mathcal{C}$ such that the following diagrams are pullbacks:

$$\begin{array}{ccc} R & \xrightarrow{\quad p_1 \quad} & \epsilon_A \\ {\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle \in_1} \\ A \times B & \xrightarrow[1_A \times \zeta]{} & A \times \mathcal{P}_s\, A \end{array} \quad \text{and} \quad \begin{array}{ccc} GR & \xrightarrow{\quad p_2 \quad} & \epsilon_{GA} \\ {\scriptstyle G\rho}\big\downarrow & & \big\downarrow{\scriptstyle \in_2} \\ GA \times GB & \xrightarrow[1_{GA} \times \zeta']{} & GA \times \mathcal{P}_s\, GA. \end{array} \tag{8}$$

Moreover, $\zeta' = \tau \circ G\zeta$.

The crucial fact relating small relations in $\mathcal{C}_G$ and with those in $\mathcal{C}$ is the following lemma.

Lemma 2.4 *If $\rho : (R, r) \rightarrowtail (A, a) \times (B, b)$ is a small relation in $\mathcal{C}_G$ such that ρ is classified by $\zeta : B \longrightarrow \mathcal{P}_s\, A$ in $\mathcal{C}$ (as above), then:*

$$\zeta \quad = \quad a^* \circ \tau \circ G\zeta \circ b. \tag{9}$$

PROOF Where ζ and ζ' are as in (8) it suffices to prove that:

$$\mathcal{C} \models \forall x : A, y : B. x \;\epsilon_A\; \zeta(y) \Leftrightarrow x \;\epsilon_A\; a^* \circ \zeta' \circ b(y).$$

Suppose given generalized elements $\delta_1 : D \longrightarrow A$ and $\delta_2 : D \longrightarrow B$.

First assume that $D \Vdash \delta_1 \;\epsilon_A\; a^* \circ \zeta' \circ b \circ \delta_2$. Then $D \Vdash a \circ \delta_1 \;\epsilon_{GA}\; \zeta' \circ b \circ \delta_2$. But by naturality of $\epsilon_{A\times B}$ we have $D \Vdash \epsilon_A \circ a \circ \delta_1 \;\epsilon_A\; \zeta \circ \epsilon_B \circ b \circ \delta_2$ which is the same as $D \Vdash \delta_1 \;\epsilon_A\; \zeta \circ \delta_2$, as required.

Alternatively, if $D \Vdash \delta_1 \;\epsilon_A\; \zeta \circ \delta_2$ then since (R, r) is a coalgebra we have $D \Vdash a \circ \delta_1 \;\epsilon_{GA}\; \zeta' \circ b \circ \delta_2$ which holds if and only if $D \Vdash \delta_1 \;\epsilon_A\; a^* \circ \zeta' \circ b \circ \delta_2$, as required. $\square$

The reader can check that we also have $\zeta = (\epsilon_A)_! \circ \zeta' \circ b$.

Proposition 2.5 *If $\mathcal{C}$ has basic class structure and G is a cartesian comonad which preserves small maps, then $\mathcal{C}_G$ also has basic class structure and the forgetful functor $U : \mathcal{C}_G \longrightarrow \mathcal{C}$ preserves small maps.*

PROOF We have already seen that $\mathcal{C}_G$ is a positive heyting category which satisfies **(S1)-(S5)**. Therefore we need only show that **(P1)** is satisfied. Let a small relation $\rho : (R, r) \rightarrowtail (A, a) \times (B, b)$ be given in $\mathcal{C}_G$ and suppose that classifying maps ζ and ζ' exist in $\mathcal{C}$ as in (8). Then we have $G\zeta \circ b : B \longrightarrow G\mathcal{P}_s\, A$. Also:

$$\begin{aligned} Ga^* \circ G\tau \circ \delta_{\mathcal{P}_s\, A} \circ G\zeta \circ b &= Ga^* \circ G\tau \circ G^2\zeta \circ \delta_B \circ b \\ &= Ga^* \circ G\tau \circ G^2\zeta \circ Gb \circ b \\ &= G\zeta \circ b, \end{aligned}$$

where the final equality is by Lemma 2.4. Because $i : E \rightarrowtail G\mathcal{P}_s\, A$ is an equalizer there exists a unique map $\eta : B \longrightarrow E$ such that $i \circ \eta = G\zeta \circ b$. Moreover, because Gi is a monomorphism this map is easily seen to be a coalgebra homomorphism $\eta : (B, b) \longrightarrow (E, e)$.

Notice that since $R \Vdash \pi_A \circ \rho \;\epsilon_A\; \zeta \circ \pi_B \circ \rho$ we have, by Lemma 2.4 that $R \Vdash \pi_A \circ \rho \;\epsilon_A\; \chi \circ \eta \circ \pi_B \circ \rho$. Therefore there exists a map $\eta' : R \longrightarrow M$ such that $1_{A\times\eta} \circ \rho = \mu \circ \eta'$. Since $G\mu$ is a monomorphism this map η' is a coalgebra homomorphism. Moreover, the following diagram commutes in $\mathcal{C}_G$:

$$\begin{array}{ccc} (R, r) & \xrightarrow{\quad \eta' \quad} & (M, m) \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ (A, a) \times (B, b) & \xrightarrow[1_{(A,a)}\times\eta]{} & (A, a) \times (E, e). \end{array} \tag{10}$$

To prove that (10) is a pullback square in $\mathcal{C}_G$ let a coalgebra (Z, z) and maps $\langle z_0, z_1\rangle : (Z, z) \longrightarrow (A, a) \times (B, b)$ and $z_2 : (Z, z) \longrightarrow (M, m)$ be given such that $\mu \circ z_2 = (1_{(A,a)} \times \eta) \circ \langle z_0, z_1\rangle$. Then, by definition of M, $Z \Vdash z_0 \; \epsilon_A \; a^* \circ \tau \circ i \circ \eta \circ z_1$. But:

$$\begin{array}{rll} Z \Vdash z_0 \; \epsilon_A \; a^* \circ \tau \circ i \circ \eta \circ z_1 & \text{iff} & Z \Vdash z_0 \; \epsilon_A \; a^* \circ \tau \circ G\zeta \circ b \circ z_1 \\ & \text{iff} & Z \Vdash z_0 \; \epsilon_A \; \zeta \circ z_1 . \end{array}$$

As such, there exists a map $\xi : Z \longrightarrow R$ such that $\rho \circ \xi = \langle z_0, z_1\rangle$ and, because μ is mono, $\eta' \circ \xi = z_2$. ξ is a coalgebra homomorphism since $G\rho$ is mono and is unique since ρ is mono.

Finally, to see that $\eta : (B, b) \longrightarrow (E, e)$ is unique suppose given $\phi : (B, b) \longrightarrow (E, e)$ and $\phi' : (R, r) \longrightarrow (M, m)$ such that the following diagram is a pullback:

$$\begin{array}{ccc} (R, r) & \xrightarrow{\phi'} & (M, m) \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \mu} \\ (A, a) \times (B, b) & \xrightarrow[1_{(A,a)} \times \phi]{} & (A, a) \times (E, e). \end{array} \tag{11}$$

We will prove that $i \circ \phi = G\zeta \circ b$ as $\eta = \phi$ will then follow from fact that i is an equalizer. First notice that the following diagram is a pullback in $\mathcal{C}$:

$$\begin{array}{ccc} R & \xrightarrow{\mu' \circ \phi'} & \epsilon_A \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \in_1} \\ A \times B & \xrightarrow[1_A \times (\chi \circ \phi)]{} & A \times \mathcal{P}_s \, A. \end{array}$$

Therefore, by uniqueness of classifying maps in $\mathcal{C}$, $\chi \circ \phi = \zeta$. But then:

$$\begin{array}{rcl} G\zeta \circ b & = & G\chi \circ G\phi \circ b \\ & = & Ga^* \circ G\tau \circ Gi \circ e \circ \phi \\ & = & Ga^* \circ G\tau \circ \delta_{\mathcal{P}_s A} \circ i \circ \phi \\ & = & i \circ \phi . \end{array}$$

Therefore, $\eta = \phi$, as required. □

We now show that the addition of a universal object (cf. Appendix A) is preserved under the coalgebra construction.

Theorem 2.6 *If $\mathcal{C}$ is a basic category of classes and G is a cartesian comonad, then $\mathcal{C}_G$ is also a basic category of classes.*

Proof By Proposition 2.5 it suffices to prove that there exists a universal object in $\mathcal{C}_G$. Let U be a universal object in $\mathcal{C}$. We claim that (GU, δ_U) is a universal object in $\mathcal{C}_G$. To see that this is so let (A, a) be given. Then we have a composite:

$$A \overset{a}{\rightarrowtail} GA \overset{G\iota_A}{\rightarrowtail} GU,$$

for some monomorphism ι_A. But $G\iota_A \circ a$ is a coalgebra homomorphism since:

$$\begin{aligned} G^2\iota_A \circ Ga \circ a &= G^2\iota_A \circ \delta_A \circ a \\ &= \delta_U \circ G\iota_A \circ a. \end{aligned}$$

□

3 The exponentiation axiom

Recall that the *exponentiation axiom* in a basic category of classes is the following (cf. [8]):

(E) If $f : C \longrightarrow D$ is a small map, then the functor $\Pi_f : \mathcal{C}/C \longrightarrow \mathcal{C}/D$ preserves small maps,

where the functor Π_f exists in any basic category of classes if f is small (cf. [1]). We will call a basic category of classes which also satisfies **(E)** a *category of classes*. Note that in any basic category of classes the small objects are exponentiable (cf. [1]), although the resulting exponentials need not be small. In order to investigate whether the exponentiation axiom is preserved by the coalgebra construction the following lemma will be useful.

Lemma 3.1 *If $\mathcal{C}$ is a basic category of classes such that the full subcategory $\mathcal{S}_\mathcal{C}$ of small objects and arrows between them is cartesian closed, then for any two small objects A and B of $\mathcal{C}$, the "small" exponential B^A in $\mathcal{S}_\mathcal{C}$ is isomorphic to the "class" exponential E in $\mathcal{C}$.*

Proof Recall that the "class" exponential E is constructed as the following subobject of $\mathcal{P}_s(A \times B)$ in $\mathcal{C}$:

$$E \quad := \quad [\![z : \mathcal{P}_s(A \times B) | \forall x : A. \exists ! y : B. (x, y) \,\epsilon_{A \times B}\, z]\!].$$

Write B^A for the exponential in $\mathcal{S}_\mathcal{C}$. We will now show that $B^A \cong E$. First, observe that, where $\mathrm{ev} : B^A \times A \longrightarrow B$ is the evaluation map:

$$G \quad := \quad [\![(x, y) : (A \times B), z : B^A | \mathrm{ev}(z, x) = y]\!]$$

is a small relation since the diagonal $B \rightarrowtail B \times B$ and the second projection $(A \times B) \times B^A \longrightarrow B^A$ are both small. Therefore there exists a unique map $\gamma : B^A \longrightarrow \mathcal{P}_s(A \times B)$ classifying G. Notice that we cannot yet conclude that γ is a monomorphism in $\mathcal{C}$ since B^A is only assumed to be an exponential in $\mathcal{S}_\mathcal{C}$. However, it is straightforward to verify using the internal language that the image of γ is isomorphic to E:

$$\begin{array}{ccc} B^A & \overset{e}{\twoheadrightarrow} & E \\ & {\scriptstyle\gamma} \searrow \quad \swarrow {\scriptstyle i} & \\ & \mathcal{P}_s(A \times B). & \end{array}$$

Since e is a cover it follows by **(S4)** that E is also a small object. Finally, we may conclude that $E \cong B^A$ since both are exponentials in $\mathcal{S}_\mathcal{C}$. □

Proposition 3.2 *If $\mathcal{C}$ is a category of classes and G is a cartesian comonad which preserves small relations, then $\mathcal{C}_G$ is also a category of classes.*

PROOF Recall from [8] that if a basic category of classes satisfies **(E)**, then the full subcategory of small objects is a Π-pretopos (a locally cartesian closed pretopos). Notice that by definition of the small map structure on $\mathcal{C}_G$ we have:

$$\mathcal{S}_{\mathcal{C}_G} \quad = \quad (\mathcal{S}_\mathcal{C})_G,$$

where the category on the right is understood to be formed with respect to the restriction of the comonad G to $\mathcal{S}_\mathcal{C}$. Since the $\mathcal{S}_\mathcal{C}$ is a Π-pretopos and the Π-pretopos structure is preserved by taking coalgebras for cartesian comonads (cf. Proposition D.4 from the appendix) it follows that $\mathcal{S}_{\mathcal{C}_G}$ is a Π-pretopos. By Lemma 3.1 it follows that if (A, a) and (B, b) are small objects, then so is $(B, b)^{(A,a)}$. We now need only verify that this holds in every slice.

Suppose given an object (D, d) of $\mathcal{C}_G$ and notice that (cf. [5]):

$$\mathcal{C}_G/(D, d) \quad \cong \quad (\mathcal{C}/D)_{G'},$$

where G' is the cartesian comonad which sends an object $f : A \longrightarrow D$ to $F_D(f)$ obtained as the following pullback in $\mathcal{C}_G$:

$$\begin{array}{ccc} F_D(f) & \longrightarrow & (D, d) \\ \big\downarrow & & \big\downarrow d \\ (GA, \delta_A) & \xrightarrow[Gf]{} & (GD, \delta_D). \end{array}$$

Moreover, G' preserves small relations since G and pullback both do. Finally, because $\mathcal{C}/D$ is a category of classes (cf. [8]), we may now apply identical reasoning to the case above to show that if f and g are small objects in $\mathcal{C}_G/(D, d)$, then so is f^g. Therefore, $\mathcal{C}_G$ satisfies **(E)**. □

4 The powerset axiom

In a basic category of classes $\mathcal{C}$, the *powerset axiom* is the following (cf. [1]):

(P2) For any object A of $\mathcal{C}$, the internal subset relation $\subseteq_A \rightarrowtail \mathcal{P}_s A \times \mathcal{P}_s A$ is a small relation.

A basic category of classes which also satisfies **(P2)** is called a *powered category of classes*. In order to extend our results to powered categories of classes it will first be instructive to investigate the internal subset relation in the category of coalgebras.

Lemma 4.1 *If $\mathcal{C}$ is a basic category of classes and G is a cartesian comonad which preserves small relations and (A, a) is an object of $\mathcal{C}_G$, then:*

$$\subseteq_{(A,a)} \quad \cong \quad (J, j),$$

where (J, j) is defined as the following pullback:

$$\begin{array}{ccc}
(J, j) & \longrightarrow & (G \subseteq_A, \delta_{\subseteq_A}) \\
{\scriptstyle \gamma_1} \downarrow & & \downarrow \\
\mathcal{P}_s\,(A, a) \times \mathcal{P}_s\,(A, a) & \xrightarrow[i \times i]{} & (G\mathcal{P}_s\,A, \delta_{\mathcal{P}_s\,A}) \times (G\mathcal{P}_s\,A, \delta_{\mathcal{P}_s\,A})
\end{array}$$

where the map $i : \mathcal{P}_s\,(A, a) \longrightarrow (G\mathcal{P}_s\,A, \delta_{\mathcal{P}_s\,A})$ is as in Section 2.

Proof Let us write (E, e) for $\mathcal{P}_s\,(A, a)$ as in Section 2. We begin by observing that:

$$U\Big([\![z : (A, a), y, z : (E, e) | z \,\epsilon_{(A,a)}\, x \Rightarrow z \,\epsilon_{(A,a)}\, y]\!] \Big)$$
$$\cong$$
$$[\![z : A, x : E, y : E | z \,\epsilon_A\, \chi(x) \Rightarrow z \,\epsilon_A\, \chi(y)]\!],$$

where U is the forgetful functor and χ is as defined in Section 2. That this is so may easily be seen using the description of the internal language of $\mathcal{C}_G$ given in Appendix C. Let us denote this subobject by $L \rightarrowtail A \times (E \times E)$.

Next we note that, by the Kripke-Joyal semantics for $\mathcal{C}_G$ given in Appendix C, for any (T, t) and generalized elements $\alpha, \beta : (T, t) \rightrightarrows (E, e)$:

$$\begin{array}{rll}
(T, t) \Vdash \alpha \subseteq_{(A,a)} \beta & \text{iff} & T \Vdash G\big(\forall z : A.L\big)[e \circ \alpha, e \circ \beta] \\
& \text{iff} & T \Vdash G\big(\subseteq_A\big)[G\chi \circ e \circ \alpha, G\chi \circ e \circ \beta] \\
& \text{iff} & T \Vdash G\big(\subseteq_A\big)[i \circ \alpha, i \circ \beta] \\
& \text{iff} & (T, t) \Vdash \big(G(\subseteq_A), \delta_{\subseteq_A}\big)[i \circ \alpha, i \circ \beta].
\end{array}$$

Therefore $(J, j) \cong \subseteq_{(A,a)}$, as required. □

Proposition 4.2 *If $\mathcal{C}$ is a powered category of classes and G is a cartesian comonad which preserves small relations, then $\mathcal{C}_G$ is also a powered category of classes.*

Proof Since $\mathcal{C}$ is a powered category of classes the relation $\subseteq_A \rightarrowtail \mathcal{P}_s\,A \times \mathcal{P}_s\,A$ is small. Because G is cartesian and preserves small relations the relation:

$$(G \subseteq_A, \delta_{\subseteq_A}) \rightarrowtail (G\mathcal{P}_s\,A, \delta_{\mathcal{P}_s\,A}) \times (G\mathcal{P}_s\,A, \delta_{\mathcal{P}_s\,A}),$$

is a small relation in $\mathcal{C}_G$. Therefore, by Lemma 4.1 and the fact that i is a small map (since it is a regular monomorphism) it follows that $\subseteq_{(A,a)}$ is also a small relation. □

5 Internal presheaves

Using the results of the preceeding sections we have as a useful corollary that, modulo one additional assumption, categories of classes and powered categories of classes are stable under the formation of internal presheaves. The reader who is unfamiliar with internal categories and internal presheaves ("category actions") is directed to [7] or [5] for definitions.

Proposition 5.1 *Let $\mathcal{C}$ be a category of classes (or a powered category of classes) and $\mathbb{D}$ an internal category in $\mathcal{C}$ with (internal) domain map $\partial_0 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0$ between the object of objects of $\mathbb{D}$ and the object of arrows of $\mathbb{D}$. If ∂_0 is a small map, then the category of internal (covariant) presheaves $\mathcal{C}^{\mathbb{D}}$ is also a category of classes (or a powered category of classes).*

PROOF Because ∂_0 is a small map in $\mathcal{C}$ the forgetful functor $V : \mathcal{C}^{\mathbb{D}} \longrightarrow \mathcal{C}/\mathbb{D}_0$ has both a left and a right adjoint. The existence of the left adjoint is standard and the right adjoint is constructed, as for toposes, using the right adjoint to pullback along ∂_0:

$$\Pi_{\partial_0} : \mathcal{C}/\mathbb{D}_1 \longrightarrow \mathcal{C}/\mathbb{D}_0,$$

which exists because ∂_0 is a small map (cf. [1]). Then $\mathcal{C}^{\mathbb{D}}$ is isomorphic to $(\mathcal{C}/\mathbb{D}_0)_G$ where G is defined as the following composite (cf. [7]):

$$\mathcal{C}/\mathbb{D}_0 \xrightarrow{\Delta_{\partial_1}} \mathcal{C}/\mathbb{D}_1 \xrightarrow{\Pi_{\partial_0}} \mathcal{C}/\mathbb{D}_0.$$

Here $\partial_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0$ is the (internal) codomain map and Δ_{∂_1} is the functor induced by pullback. G is cartesian since it is a right adjoint and it preserves small maps since both Δ_{∂_1} and Π_{∂_0} do (here the exponentiation axiom **(E)** is required in $\mathcal{C}$). Because the structure of categories of classes and powered categories of classes is preserved under slicing it follows from Proposition 3.2 and Proposition 4.2 that $\mathcal{C}^{\mathbb{D}}$ is a category of classes if $\mathcal{C}$ is and a powered category of classes if $\mathcal{C}$ is, respectively. □

Observe that here small maps in $\mathcal{C}^{\mathbb{D}}$ are just those that are pointwise small in $\mathcal{C}$. Also, by using instead the internal opposite category $\mathbb{D}^{\mathrm{op}}$, the analogous result for the usual internal (contravariant) presheaves is obtained.

Proposition 5.1 is related to a result of Gambino [4]. Specifically, in [4] it is shown that where $\mathcal{C}_*$ is the category consisting of classes and functions between them in a sufficiently strong—one which satisfies the strong collection, exponentiation and set induction axioms in addition to basic set forming axioms—constructive set theory **S**, then internal presheaves in $\mathcal{C}_*$ provide a model of **S**. Set theories satisfying the powerset axiom and powered categories of classes are not considered at all in [4]. Some of the key differences between Proposition 5.1 and the result of [4] may then be summarized as follows:

- In [4] the category $\mathcal{C}_*$ is fixed and possesses significant—Gambino's axiom **(S6)** in particular—additional structure beyond that provided by a category of classes.

- In Proposition 5.1 the category $\mathcal{C}$ is allowed to vary over categories of classes, but the resulting category $\mathcal{C}^{\mathbb{D}}$ need not be a model of the set theory **S**.

6 Comma categories

A second corollary to the foregoing results is that the three kinds of class category under consideration are stable under the formation of comma categories. This has immediate applications to logic in that we may then infer that the set theories modelled algebraically in these three classes of structure have the disjunction and (definable) existence properties, using a modification of the familiar Freyd cover or Artin-Wraith gluing argument (cf. [3] and [5]). These and other logical applications of these results will be taken up in more detail in a future paper: [9].

Lemma 6.1 *If $\mathcal{C}$ and $\mathcal{D}$ are basic categories of classes (or categories of classes or powered categories of classes), then so is $\mathcal{C} \times \mathcal{D}$.*

Proof The positive heyting structure of $\mathcal{C} \times \mathcal{D}$ is computed pointwise. A map $(f, f') : (C, D) \longrightarrow (C', D')$ in $\mathcal{C} \times \mathcal{D}$ is defined to be small if f is small in $\mathcal{C}$ and f' is small in $\mathcal{D}$. Powerobjects and membership relations are also defined pointwise. It is straightforward to verify that the claim then holds. □

In order to simplify the following we will adopt the convention of writing $A_0 := \pi_{\mathcal{C}}(A)$ and $A_1 := \pi_{\mathcal{D}}(A)$ where A is an object of $\mathcal{C} \times \mathcal{D}$. Similarly for maps $f : A \longrightarrow B$ in $\mathcal{C} \times \mathcal{D}$.

Proposition 6.2 *Let $\mathcal{C}$ and $\mathcal{D}$ be basic categories of classes. If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a cartesian functor which preserves small maps, then the comma category $(D \downarrow F)$ is a basic category of classes and the functor $\pi_{\mathcal{C}} : (D \downarrow F) \longrightarrow \mathcal{C}$ is logical. The same holds if $\mathcal{C}$ and $\mathcal{D}$ are categories of classes or powered categories of classes.*

Proof As is well known, if F is cartesian we have:

$$(D \downarrow F) \quad \simeq \quad (\mathcal{C} \times \mathcal{D})_G,$$

where G is the cartesian comonad given by the endofunctor

$$G(C, D) \quad := \quad (C, D \times FC).$$

Since F preserves small maps so does G and, by Theorem 2.6 $(\mathcal{C} \times \mathcal{D})_G$ is a basic category of classes. Similarly, it is a category of classes by Proposition 3.2 or a powered category of classes by Proposition 4.2.

To see that $\pi_{\mathcal{C}}$ is logical note that small maps are trivially preserved by $\pi_{\mathcal{C}}$. As such, we need only show that powerobjects, universal objects and universal quantification are preserved by $\pi_{\mathcal{C}}$. First, notice that if (A, a) is a coalgebra in $\mathcal{C}_G$, then:

$$\begin{aligned} \tau_0 &= 1_{\mathcal{P}_s A_0}, \text{ and} \\ a_0 &= 1_{A_0}, \end{aligned}$$

where $\tau : G\mathcal{P}_s\, A \longrightarrow \mathcal{P}_s\, GA$ is the comparison map in $\mathcal{C} \times \mathcal{D}$. The first equality follows from the fact that classifying maps are built pointwise in $\mathcal{C} \times \mathcal{D}$ and the second equality is by the definition of the counit $\epsilon_A : GA \longrightarrow A$ as $(1_{A_0}, \pi_{A_1})$.

Now, since $\delta_A = (1_{A_0}, \langle 1_{A_1 \times FA_0}, \pi_{FA_0} \rangle)$ we have that $\pi_C(\mathcal{P}_s\,(A, a)) = \mathcal{P}_s\, A_0$ when (A, a) is a coalgebra. The universal object is also preserved by the description of δ. Finally, the only thing that remains is to verify that universal quantification is preserved. This is also by the fact that $a_0 = 1_{A_0}$ where (A, a) is a coalgebra. $\square$

A Appendix: Axioms for categories of classes

In this appendix we summarize the axioms and definitions for categories of classes. In all of the following the ambient category $\mathcal{C}$ is assumed to be a positive heyting category. Further details may be found in [1], [2] or [8].

Definition A.1 $\mathcal{C}$ has a *system of small maps* $\mathcal{S}$ if there exists a collection $\mathcal{S}$ of maps from $\mathcal{C}$ called the *small maps* such that:

(S1) $\mathcal{S}$ is closed under composition and all identity arrows are in $\mathcal{S}$.

(S2) If the following is a pullback diagram:

$$\begin{array}{ccc} C' & \xrightarrow{g'} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow[g]{} & D \end{array}$$

and f is in $\mathcal{S}$, then f' is in $\mathcal{S}$.

(S3) All diagonals $\Delta : C \longrightarrow C \times C$ are contained in $\mathcal{S}$.

(S4) If e is a cover, g is in $\mathcal{S}$ and the diagram:

$$\begin{array}{ccc} C & \xrightarrow{e} & D \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & A & \end{array}$$

commutes, then f is in $\mathcal{S}$.

(S5) If $f : C \longrightarrow A$ and $g : D \longrightarrow A$ are in $\mathcal{S}$, then so is the copair $[f, g] : C + D \longrightarrow A$.

If A is an object of $\mathcal{C}$ such that the canonical map $A \longrightarrow 1$ is small, then we say that A is a *small object*. If $\rho : R \rightarrowtail A \times B$ is a relation such that $\pi_B \circ \rho$ is a small map, then we say that ρ is a *small relation*. A subobject $S \rightarrowtail A$ is a *small subobject* if its domain is a small object.

Definition A.2 $\mathcal{C}$ is said to have *basic class structure* if it has a system of small maps satisfying:

(P1) For each object C of $\mathcal{C}$ there exists a *power object* $\mathcal{P}_s(C)$ and a small *membership relation* $\epsilon_C \rightarrowtail C \times \mathcal{P}_s(C)$ such that, for any D and small relation $R \rightarrowtail C \times D$, there exists a unique map $\rho : D \longrightarrow \mathcal{P}_s C$ such that the square:

$$\begin{array}{ccc} R & \longrightarrow & \epsilon_C \\ \downarrow & & \downarrow \\ C \times D & \xrightarrow[1_C \times \rho]{} & C \times \mathcal{P}_s C \end{array}$$

is a pullback.

Definition A.3 A category $\mathcal{C}$ with basic class structure is a *basic category of classes* if it has a universal object. I.e., if it satisfies:

(U) There exists an object U in $\mathcal{C}$ such that, for any object A of $\mathcal{C}$, there exists a monomorphism $A \rightarrowtail U$.

Recall from [1] that if $\mathcal{C}$ has basic class structure and $f : A \longrightarrow B$ is a small map, then there exists a right adjoint $\Pi_f : \mathcal{C}/A \longrightarrow \mathcal{C}/B$ to the pullback functor $\Delta_f : \mathcal{C}/B \longrightarrow \mathcal{C}/A$.

Definition A.4 A basic category of classes $\mathcal{C}$ is said to be a *category of classes* if it also satisfies:

(E) If $f : A \longrightarrow B$ is a small map, then Π_f preserves small maps.

Definition A.5 A basic category of classes $\mathcal{C}$ is said to be a *powered category of classes* if it also satisfies:

(P2) For any A, the internal subset relation $\subseteq_A \rightarrowtail \mathcal{P}_s\, A \times \mathcal{P}_s\, A$ is a small relation.

B Appendix: Heyting structure of $\mathcal{C}_G$

We begin by reviewing some of the basic properties of the category $\mathcal{C}_G$. There is an evident forgetful functor $U : \mathcal{C}_G \longrightarrow \mathcal{C}$ which sends a coalgebra (A, a) to the object A and a map $f : (A, a) \longrightarrow (B, b)$ to $f : A \longrightarrow B$. There is also a "cofree coalgebra" functor $F : \mathcal{C} \longrightarrow \mathcal{C}_G$ which sends A to the "cofree coalgebra" (GA, δ_A) and a map $f : A \longrightarrow B$ to Gf. Moreover, if (A, a) is a coalgebra, then $a : (A, a) \longrightarrow (GA, \delta_A)$ by definition.

Proposition B.1 *Where U and F are as above, $U \dashv F$.*

Proof The following isomorphism is easily established:

$$\mathrm{Hom}_{\mathcal{C}}\big(U(A, a), B\big) \quad \cong \quad \mathrm{Hom}_{\mathcal{C}_G}\big((A, a), FB\big),$$

by sending $f : A \longrightarrow B$ to $Gf \circ a$ and $g : (A, a) \longrightarrow FB$ to $\varepsilon_B \circ g$. □

Corollary B.2 *The forgetful functor U preserves (small) colimits.*

Proposition B.3 *If G is cartesian, then the forgetful functor $U : \mathcal{C}_G \longrightarrow \mathcal{C}$ creates finite limits.*

Proof Let a functor $D : \mathcal{I} \longrightarrow \mathcal{C}_G$ with finite $\mathcal{I}$ be given and suppose the limit L of $U \circ D$ together with its cone $p_i : L \longrightarrow D_i$ exists in $\mathcal{C}$. Since U is the forgetful functor there is, for each i of $\mathcal{I}$, a coalgebra (D_i, d_i) in $\mathcal{C}_G$. As such, L together with the maps $(d_i \circ p_i : L \longrightarrow GD_i)_i$ forms a cone for $G \circ U \circ D$. Since G is cartesian there exists a unique map $l : L \longrightarrow GL$ such that $Gp_i \circ l = d_i \circ p_i$

for each i. We claim that (L, l) is a coalgebra and is, moreover, the limit of D in $\mathcal{C}_G$.

The equation $\varepsilon_L \circ l = 1_L$ follows from the fact that L is a limit and that:

$$\begin{array}{rcl} p_i \circ \varepsilon_L \circ l & = & \varepsilon_{D_i} \circ G p_i \circ l \\ & = & \varepsilon_{D_i} \circ d_i \circ p_i \\ & = & p_i, \end{array}$$

for each i. Similarly, that $\delta_L \circ l = Gl \circ l$ follows from the fact that $G^2 L$ is a limit and that:

$$G^2 p_i \circ \delta_L \circ l \quad = \quad G^2 p_i \circ Gl \circ l,$$

for each i. Clearly each p_i is a coalgebra homomorphism.

Finally, given any cone $q_i : (K, k) \longrightarrow (D_i, d_i)$ for D in $\mathcal{C}_G$ we obtain a corresponding cone $q_i : K \longrightarrow D_i$ in $\mathcal{C}$. Therefore there exists a unique map $\zeta : K \longrightarrow L$ in $\mathcal{C}$ such that $p_i \circ \zeta = q_i$ for each i. ζ is in fact as a coalgebra homomorphism since GL is a limit in $\mathcal{C}$ and:

$$\begin{array}{rcl} G p_i \circ l \circ \zeta & = & d_i \circ p_i \circ \zeta \\ & = & d_i \circ q_i \\ & = & G q_i \circ k \\ & = & G p \circ G\zeta \circ k, \end{array}$$

for each i. Uniqueness of $\zeta : (K, k) \longrightarrow (L, l)$ is an immediate consequence of the uniqueness of ζ in $\mathcal{C}$. □

Corollary B.4 *If $\mathcal{C}$ and G are cartesian, then so are $\mathcal{C}_G$ and $U : \mathcal{C}_G \longrightarrow \mathcal{C}$.*

Recall the following highly useful fact about coalgebras.

Lemma B.5 *Where $\mathcal{C}$ and G are cartesian, if (A, a) is a coalgebra and $m : D \rightarrowtail A$ is given, then there exists at most one map $d : D \longrightarrow GD$ such that $a \circ m = Gm \circ d$. Moreover, if the following diagram commutes:*

$$\begin{array}{ccc} D & \xrightarrow{d} & GD \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Gm} \\ A & \xrightarrow[a]{} & GA \end{array}$$

where m is a monomorphism and (A, a) is a coalgebra, then (D, d) is also a coalgebra and the square is a pullback.

Proof Cf. [7]. □

Using a similar argument to that given in the proof of Proposition B.3 we establish a related fact for colimits.

Proposition B.6 *$U : \mathcal{C}_G \longrightarrow \mathcal{C}$ creates all (small) colimits.*

PROOF Let a small diagram $D : \mathcal{I} \longrightarrow \mathcal{C}_G$ be given. Suppose the colimit $p_i : D_i \longrightarrow L$ of $U \circ D$ exists in $\mathcal{C}$. Then $(Gp_i \circ d_i : D_i \longrightarrow GL)_i$ is a cocone for $U \circ D$ in $\mathcal{C}$. Therefore there exists a unique map $l : L \longrightarrow GL$ such that $Gp_i \circ d_i = l \circ p_i$ for each i. Because L is a colimit in $\mathcal{C}$ and:

$$\varepsilon_L \circ l \circ p_i = \varepsilon_L \circ Gp_i \circ d_i = p_i \circ \varepsilon_D \circ d_i = p_i,$$

for each i it follows that $\varepsilon_L \circ l = 1_L$. Similarly we have that $\delta_L \circ l = Gl \circ l$. Therefore (L, l) is a coalgebra and $p_i : (D_i, d_i) \longrightarrow (L, l)$ is a cocone.

Given any other cocone $q_i : (D_i, d_i) \longrightarrow (K, k)$ there exists a unique map $\zeta : L \longrightarrow K$ in $\mathcal{C}$ such that $\zeta \circ p_i = q_i$ for each i. Using the universal property of colimits it is easily seen that ζ is a coalgebra homomorphism. Uniqueness of ζ in $\mathcal{C}_G$ is then a direct consequence of its uniqueness in $\mathcal{C}$. □

Lemma B.7 *If $\mathcal{C}$ is regular and G is cartesian, then any map $e : (A, a) \longrightarrow (B, b)$ is a regular epimorphism in $\mathcal{C}_G$ if and only if $e : A \longrightarrow B$ is a regular epimorphism in $\mathcal{C}$.*

PROOF Suppose e is a regular epimorphism in $\mathcal{C}_G$. Then there exists an object (D, d) and arrows h, k such that the following is a coequalizer diagram:

$$(D, d) \overset{h}{\underset{k}{\rightrightarrows}} (A, a) \xrightarrow{e} (B, b).$$

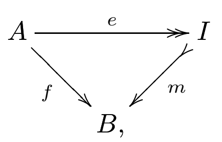
By Corollary B.2 e coequalizes h and k in $\mathcal{C}$ also. Therefore e is a regular epimorphism in $\mathcal{C}$.

Suppose instead that e is a regular epimorphism in $\mathcal{C}$. As such, since $\mathcal{C}$ is regular, e coequalizes its own kernel pair $k_0, k_1 : K \rightrightarrows A$. But, by the description of finite limits in $\mathcal{C}_G$, there exists a map $k : K \longrightarrow GK$ so that (K, k) is a coalgebra and $k_0, k_1 : (K, k) \rightrightarrows (A, a)$. It is then straightforward to verify that e coequalizes k_0 and k_1 in $\mathcal{C}_G$. □

Proposition B.8 *If $\mathcal{C}$ is regular and G cartesian, then $\mathcal{C}_G$ is also regular. Moreover, U is a regular functor.*

PROOF Since G is cartesian it follows from Corollary B.4 that $\mathcal{C}_G$ is cartesian.

To see that $\mathcal{C}_G$ has image factorizations let $f : (A, a) \longrightarrow (B, b)$ in $\mathcal{C}_G$ be given. In $\mathcal{C}$ we may factor f as follows:



where e is a regular epimorphism and m is a monomorphism. Similarly, we may factor Gf as follows:

$$\begin{array}{ccc} GA & \xrightarrow{e'} & I' \\ & {\scriptstyle Gf}\searrow \quad \swarrow{\scriptstyle m'} & \\ & GB, & \end{array}$$

where e' is a regular epimorphism and m' is a monomorphism. Therefore, the following diagram commutes in $\mathcal{C}$:

$$\begin{array}{ccc} GA & \xrightarrow{e'} & I' \\ {\scriptstyle Ge}\downarrow & & \downarrow{\scriptstyle m'} \\ GI & \xrightarrow[Gm]{} & GB. \end{array}$$

Since e' is a regular epimorphism there exists a unique "diagonal filler" $g : I' \longrightarrow GI$ such that $g \circ e' = Ge$ and $Gm \circ g = m'$. Similarly, since $m' \circ e' \circ a = b \circ m \circ e$, there exists a unique map $g' : I \longrightarrow I'$ such that $m' \circ g' = b \circ m$ and $g' \circ e = e' \circ a$. Define $i : I \longrightarrow GI$ to be the composite $g \circ g'$. Then $Gm \circ i = b \circ m$ and (I, i) is a coalgebra by Lemma B.5. Because e is a coalgebra homomorphism it follows by Lemma B.7 that e is a regular epimorphism in $\mathcal{C}_G$. Since m is a monomorphism in $\mathcal{C}_G$ we have shown that f may be factored as a regular epimorphism followed by a monomorphism.

To see that this factorization is unique observe that if $f = m'' \circ e''$ in $\mathcal{C}_G$ with e'' a regular epimorphism and m'' a monomorphism, then e'' is a regular epimorphism in $\mathcal{C}$ and m'' is a monomorphism in $\mathcal{C}$. By uniqueness of the factorization $f = m \circ e$ in $\mathcal{C}$ we have $m'' = m$ and $e'' = e$.

Next, suppose $e : (A, a) \longrightarrow (B, b)$ is a regular epimorphism in $\mathcal{C}_G$ and let $f : (D, d) \longrightarrow (B, b)$ be given to show that the resulting map $e' : (A', a') \longrightarrow (D, d)$ obtained by pulling e back along f is also a regular epimorphism. By the description of pullbacks in $\mathcal{C}_G$ we have that $e' : A' \longrightarrow D$ is the pullback of e along f in $\mathcal{C}$. But, by Lemma B.7, e is regular in $\mathcal{C}$ so that e' is also a regular epimorphism. Again, by Lemma B.7 e' is regular in $\mathcal{C}_G$.

Finally, U is a regular functor by Corollary B.4 and Lemma B.7. □

Notice that when $\mathcal{C}$ and G are cartesian there is a forgetful functor:

$$U_{(A,a)} : \mathrm{Sub}_{\mathcal{C}_G}(A, a) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(A),$$

for each coalgebra (A, a), which sends a subobject $m : (D, d) \rightarrowtail (A, a)$ to $m : D \rightarrowtail A$. Moreover, when $f : (A, a) \longrightarrow (B, b)$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Sub}_{\mathcal{C}_G}(B, b) & \xrightarrow{U_{(B,b)}} & \mathrm{Sub}_{\mathcal{C}}(B) \\ {\scriptstyle f_G^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Sub}_{\mathcal{C}_G}(A, a) & \xrightarrow[U_{(A,a)}]{} & \mathrm{Sub}_{\mathcal{C}}(A) \end{array} \qquad (12)$$

where f_G^* denotes the pullback functor from $\mathcal{C}_G$.

Lemma B.9 *If (A,a) is a coalgebra in $\mathcal{C}_G$, then there exists a functor $F_{(A,a)}$: $\mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}_G}(A,a)$ such that $U_{(A,a)} \dashv F_{(A,a)}$. Moreover:*

$$F_{(A,a)} \circ U_{(A,a)} m \cong m,$$

for any $m : (B,b) \rightarrowtail (A,a)$. In particular, $U_{(A,a)}$ is full and faithful.

Proof Given a subobject $m : D \rightarrowtail A$ in $\mathcal{C}$ we may form the pullback of Gm along a as indicated in the following diagram:

$$\begin{array}{ccc} P & \overset{p_1}{\rightarrowtail} & A \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle a} \\ GD & \underset{Gm}{\rightarrowtail} & GA. \end{array} \tag{13}$$

Since G is cartesian we have that GP is also the pullback of Ga along G^2m. As such, there are maps $\delta_D \circ p_0 : P \longrightarrow G^2D$ and $a \circ p_1 : P \longrightarrow GA$. Moreover:

$$Ga \circ a \circ p_1 = \delta_A \circ a \circ p_1 = \delta_A \circ Gm \circ p_0 = G^2m \circ \delta_D \circ p_0.$$

Therefore there exists a unique map $p : P \longrightarrow GP$ such that $Gp_0 \circ p = \delta_D \circ p_0$ and $Gp_1 \circ p = a \circ p_1$. By Lemma B.5, (P,p) is a coalgebra and $p_1 : (P,p) \rightarrowtail (A,a)$ is a subobject. Let $F_{(A,a)}(m : D \rightarrowtail A) := (P,p)$. It is straightforward to verify that this is a functor.

To see that $U_{(A,a)} \dashv F_{(A,a)}$ let subobjects $m : D \rightarrowtail A$ and $n : B \rightarrowtail A$ be given such that $n = U_{(A,a)}\big(n : (B,b) \rightarrowtail (A,a)\big)$. Suppose there exists a map $g : B \longrightarrow D$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \overset{g}{\rightarrowtail} & D \\ & {\scriptstyle n}\searrow \quad \swarrow{\scriptstyle m} & \\ & A. & \end{array}$$

Write $p_1 : (P,p) \rightarrowtail (A,a)$ for $F_{(A,a)}(m : D \rightarrowtail A)$. By definition, (P,p) is as described above in (13). Because $Gm \circ Gg \circ b = a \circ n$ there exists, by the definition of P, a unique map $g' : B \longrightarrow P$ in $\mathcal{C}$ such that $p_0 \circ g' = Gg \circ b$ and $p_1 \circ g' = n$. Using the fact that Gp_0 is monomorphic it is easy to verify that g' is a coalgebra homomorphism $(B,b) \rightarrowtail (P,p)$ such that $p_1 \circ g' = n$, as required.

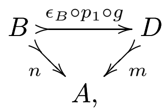
Alternatively, let $m : D \rightarrowtail A$ in $\mathcal{C}$ and $n : (B,b) \rightarrowtail (A,a)$ in $\mathcal{C}_G$ be given. Write $p_1 : (P,p) \rightarrowtail (A,a)$ for $F_{(A,a)}(m)$ and suppose (P,p) is as described in (13). Assume the following diagram commutes:

$$\begin{array}{ccc} (B,b) & \overset{g}{\rightarrowtail} & (P,p) \\ & {\scriptstyle n}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & (A,a). & \end{array}$$

Then, in $\mathcal{C}$, we have $\epsilon_B \circ p_0 \circ g : B \longrightarrow D$. Moreover:

$$m \circ \epsilon_D \circ p_0 \circ g = \epsilon_A \circ a \circ n = n.$$

I.e., the following diagram commutes:



as required.

Finally, $F_{(A,a)} \circ U_{(A,a)}(m) \cong m$ for any subobject $m : (B,b) \rightarrowtail (A,a)$ in $\mathcal{C}_G$ by Lemma B.5. □

Proposition B.10 *If $\mathcal{C}$ is a positive heyting category and G is cartesian, then $\mathcal{C}_G$ is also a positive heyting category.*

Proof By Proposition B.6, $\mathcal{C}_G$ has finite coproducts. These finite coproducts are disjoint and stable since pullbacks are computed in $\mathcal{C}$.

Let a map $f : (A,a) \longrightarrow (B,b)$ in $\mathcal{C}_G$ be given to show that the pullback functor $f^* : \mathrm{Sub}_{\mathcal{C}_G}(B,b) \longrightarrow \mathrm{Sub}_{\mathcal{C}_G}(A,a)$ has a right adjoint f_*^G. Here the superscript G is used to distinguish this from the dual image functor f_* in $\mathcal{C}$. Similarly, we will here write f_G^* for the pullback functor in $\mathcal{C}_G$.

Specifically, f_*^G is defined as the following composite:

$$f_*^G \quad := \quad F_{(B,b)} \circ f_* \circ U_{(A,a)}.$$

To see that this definition works suppose given subobjects $n : (S,s) \rightarrowtail (B,b)$ and $m : (D,d) \rightarrowtail (A,a)$. Then:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sub}_{\mathcal{C}_G}(B,b)}(n, f_*^G(m)) &\cong \mathrm{Hom}_{\mathrm{Sub}_{\mathcal{C}}(B)}(U_{(B,b)}(n), f_* \circ U_{(A,a)}(m)) \\
&\cong \mathrm{Hom}_{\mathrm{Sub}_{\mathcal{C}}(A)}(f^* \circ U_{(B,b)}(n), U_{(A,a)}(m)) \\
&\cong \mathrm{Hom}_{\mathrm{Sub}_{\mathcal{C}}(A)}(U_{(A,a)} \circ f_G^*(n), U_{(A,a)}(m)) \\
&\cong \mathrm{Hom}_{\mathrm{Sub}_{\mathcal{C}_G}(A,a)}(f_G^*(n), m),
\end{aligned}$$

where the penultimate isomorphism is by (12) and the final isomorphism is by Lemma B.9. □

C Appendix: Kripke-Joyal semantics of $\mathcal{C}_G$

The Kripke-Joyal semantics is one of the principal tools for reasoning logically about the internal language of a category. In the following we will summarize the clauses of the Kripke-Joyal semantics for categories of the form $\mathcal{C}_G$ where $\mathcal{C}$ is a positive Heyting category and G is a Cartesian functor. The reader who is unfamiliar with the usual Kripke-Joyal semantics is referred to [7].

Lemma C.1 *Let a subobject $m : (A,a) \rightarrowtail (D,d)$ in $\mathcal{C}_G$ be given. Then, for any $\tau : (T,t) \longrightarrow (D,d)$ in $\mathcal{C}_G$:*

$$(T,t) \Vdash (A,a)[\tau] \text{ in } \mathcal{C}_G \quad \text{if and only if} \quad T \Vdash A[\tau] \text{ in } \mathcal{C}.$$

Proof For the non-trivial direction suppose given $f : T \longrightarrow A$ such that $m \circ f = \tau$. We need to show that f is in fact a coalgebra homomorphism. But:

$$Gm \circ a \circ f = b \circ m \circ f = b \circ \tau = G\tau \circ t = Gm \circ Gf \circ t.$$

Since G is Cartesian we have $a \circ f = Gf \circ t$, as required. □

Notice that the right hand side of the equivalence above is an abbreviation for:

$$T \Vdash U(A, a)[\tau]$$

and, as such, the characterization of the Kripke-Joyal semantics depends on which logical properties are preserved by U.

Using the properties established in Appendix B we obtain the following clauses for the Kripke-Joyal semantics in $\mathcal{C}_G$.

Proposition C.2 *Let subobjects $m : (A, a) \rightarrowtail (D, d)$, $m' : (B, b) \rightarrowtail (D', d')$ and maps $\tau : (T, t) \longrightarrow (D, d)$ and $\tau' : (T, t) \longrightarrow (D', d')$ be given. Then:*

1. $(T, t) \Vdash (A, a)[\tau] \wedge (B, b)[\tau']$ *if and only if* $T \Vdash A[\tau]$ *and* $T \Vdash B[\tau']$.
2. $(T, t) \Vdash (A, a)[\tau] \vee (B, b)[\tau']$ *if and only if* $T \Vdash A[\tau] \vee B[\tau']$.
3. $(T, t) \Vdash (A, a)[\tau] \Rightarrow (B, b)[\tau']$ *if and only if* $T \Vdash A[\tau] \Rightarrow B[\tau']$.
4. $(T, t) \Vdash \neg(A, a)[\tau]$ *if and only if* $T \Vdash \neg A[\tau]$.

Moreover, given $m : (D, d) \rightarrowtail (A, a) \times (B, b)$ and $\tau : (T, t) \longrightarrow (B, b)$ we have the following clauses for the quantifiers:

5. $(T, t) \Vdash \exists z : (A, a).(D, d)[\tau, z]$ *if and only if* $T \Vdash \exists z : A.D[\tau, z]$.
6. $(T, t) \Vdash \forall z : (A, a).(D, d)[\tau, z]$ *if and only if* $T \Vdash G\big(\forall x : A.D\big)[b \circ \tau]$.

Proof (1) and (2) are by Lemma C.1 together with the fact that the forgetful functor $U : \mathcal{C}_G \longrightarrow \mathcal{C}$ preserves meets and joins. For (3) we recall that:

$$\begin{aligned} (A, a) \Rightarrow (B, b) &\cong m^G_*\big((A, a) \wedge (B, b) \rightarrowtail (A, a)\big) \\ &\cong F_{(D,d)\times(D',d')}\big(A \Rightarrow B\big). \end{aligned}$$

Using this description of $(A, a) \Rightarrow (B, b)$ it is straightforward to verify (3). (4) is then a direct consequence of (3) and the fact that U preserves the initial object.

(5) is by Proposition B.8. Finally, (6) is by the description of dual images given in Proposition B.10. □

D Local cartesian closure and Π-pretopos structure

The first result of this section is well known and the proof can be found in, e.g., Johnstone [5].

Lemma D.1 *If $\mathcal{C}$ is a locally cartesian closed category and G is a cartesian comonad, then $\mathcal{C}_G$ is locally cartesian closed.*

Recall from [8] that a Π-pretopos is a locally cartesian closed pretopos. We will now show that the structure of Π-pretoposes is also preserved by taking coalgebras for a cartesian comonad. Note that in this section no small map or powerobject axioms are assumed to hold in any of the categories under consideration unless explicitly mentioned. Throughout $\mathcal{C}$ is assumed to be a cartesian category and G a cartesian comonad.

Lemma D.2 *If $\partial_0, \partial_1 : (R, r) \rightrightarrows (A, a)$ is an equivalence relation in $\mathcal{C}_G$, then it is an equivalence relation in $\mathcal{C}$ (i.e., the forgetful functor U preserves equivalence relations).*

Proof By Corollary B.4 from Appendix B. □

Lemma D.3 *If $\mathcal{C}$ is a heyting pretopos, then so is $\mathcal{C}_G$.*

Proof By Proposition B.10 from Appendix B, $\mathcal{C}_G$ is a positive heyting category. To see that $\mathcal{C}_G$ is effective let an equivalence relation $\partial_0, \partial_1 : (R, r) \rightrightarrows (A, a)$ in $\mathcal{C}_G$ be given. By the lemma ∂_0, ∂_1 is an equivalence relation in $\mathcal{C}$. Therefore there exists a coequalizer $f : A \twoheadrightarrow Q$ as indicated in the following diagram:

$$R \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} A \xrightarrow{f} Q,$$

and such that ∂_0, ∂_1 are the kernel pair of f in $\mathcal{C}$. Since the forgetful functor creates colimits by Proposition B.6 from Appendix B there exists a map $q : Q \longrightarrow GQ$ such that (Q, q) is a coalgebra and f is the coequalizer of ∂_0, ∂_1 in $\mathcal{C}_G$. Moreover, ∂_0, ∂_1 is the kernel pair of q in $\mathcal{C}_G$. □

Proposition D.4 *If $\mathcal{C}$ is a Π-pretopos, then so is $\mathcal{C}_G$.*

Proof $\mathcal{C}_G$ is a heyting pretopos by Lemma D.3 and a locally cartesian closed category by Lemma D.1. □

References

[1] S. Awodey, C. Butz, A. Simpson, and T. Streicher. Relating set theories, toposes and categories of classes. In preparation, 2004. A preliminary version is available as Carnegie Mellon University Department of Philosophy Technical Report CMU-PHIL-146, June 2003.

[2] S. Awodey and M. A. Warren. Predicative algebraic set theory. *Theory and Applications of Categories*, 15:1–39, 2005.

[3] P. Freyd. Aspects of topoi. *Bulletin of the Australian Mathematical Society*, 7:1–76, 1972.

[4] N. Gambino. Presheaf models of constructive set theories. In L. Crosilla and P. Schuster, editors, *From Sets and Types to Topology and Analysis.* Oxford University Press, 2005.

[5] P. T. Johnstone. *Sketches of an Elephant*, volume 1. Oxford University Press, Oxford, 2003.

[6] A. Joyal and I. Moerdijk. *Algebraic Set Theory.* Cambridge University Press, Cambridge, 1995.

[7] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic.* Springer-Verlag, Berlin, 1992.

[8] M. A. Warren. Predicative categories of classes. Master's thesis, Carnegie Mellon University, 2004.

[9] M. A. Warren. Disjunction and existence in algebraic set theory. *In preparation*, 2006.